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·基础理论研究·

Construction of multiscaling functions with dilation factor $a=3$

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Abstract: A method for constructing orthogonal symmetric compactly supported multiscaling functions with dilation factor 3 is given. First, compactly supported orthogonal complex-valued uni-scaling function with dilation factor 3 is constructed by using real uni-scaling function. Then by using the above obtaining complex ones, multiscaling functions with dilation factor 3 which are orthogonal symmetric are constructed. Finally, some design examples are given.

Key words: compact support; orthogonal; scaling function; dilation factor

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0 Introduction

Recently, multiwavelets construction generated by a finite collection of scaling functions (i.e., multiscaling functions) have been studied. The main motivation for multiwavelets is that they can simultaneously possess desirable properties such as symmetry, orthogonality, and shorter support for a given approximation order, which are not possible in any real-valued scalar wavelet. One of the earliest and most popular used multiwavelets with multiplicity 2 is the GHM multiwavelets which was constructed by Geronimo et al using fractal interpolation. The multiscaling functions of the GHM multiwaveltes are both symmetric and orthogonal. Later, by imposing Hermite interpolation conditions, CHUI and LIAN[1] constructed symmetric and anti-symmetric orthonormal multiwavelets.

It is well-known that the multiscaling function with dilation factor $a=2$ play an essential role in the construction of multiwavelets. So there is considerable literature voted to the construction of multiscaling function with dilation factor $a=2$. For $a>2$ case, there are also some literature discussed the construction of wavelet. For example, for uni-wavelet case, LIAN[2] constructed symmetric compactly supported orthogonal scaling functions with scaling factor $a=3$ and the two corresponding compactly supported orthogonal wavelets, one of which is symmetric and the other antisymmetric; for multiwavelts case, the literature [3] also discussed the construction of multiwavelets with dilation factor $a>2$. However, the construction of having certain properties multiwavelets with dilation factor $a>2$ is not simple. In this paper, we present a method to construct symmetric multiscaling functions $\Phi(x)$ with dilation factor $a=3$ from symmetric compactly supported orthogonal uni-scaling functions while complex-valued uni-multiscaling function serving as a link, which is easily implementable and different from the existing methods.

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Biography: TIAN Hua (1977-), male, native of Jishou, Hunan province, now studying as a master student in Xi'an Jiaotong University, specialization in wavelet analysis.

1 Basic theory

Let $\Psi(x) = (\psi_1, \cdots, \psi_{2r})^T$ be a multiwavelets with dilation factor 3 if $\psi_l \in L^2$, $1 \le l \le 2r$, and the family $\{\psi_{l;j,k} : 1 \le l \le 2r : j, k \in Z\}$ constitutes a Riesz basis of L^2.

For constructing multiwavelets that generate MRA of L^2, as a usual approach, scaling functions need to be constructed. We will consider $\Phi(x) = (\varphi_1, \varphi_2, \cdots, \varphi_r)^T$ satisfying two scale matrix equations:

$$\Phi(x) = \sum_{k \in Z} P_k \Phi(3x - k) \qquad (1)$$

where $P_k, k \in Z$ are $r \times r$ real matrixs $\Phi(x)$ satisfying (1) will be called a multiscaling function with dilation factor 3 and multiplicity r (when $r = 1$, $\varphi(x)$ will be called a uni-scaling function). If the family

$$\{\varphi_l(\cdot - k) : 1 \le l \le r : k \in Z\} \qquad (2)$$

constitues a Riesz basis of its L^2-closure. A multiscaling fuction $\Phi(x)$ is orthogonal if the family (2) is also. For convenience, the function φ_l, $1 \le l \le r$ will also called scaling functins. It is clear from (1) that

$$\hat{\Phi}(w) = P(z)\hat{\Phi}\left(\frac{w}{3}\right) \qquad (3)$$

where $z = e^{-\frac{iw}{3}}$ and $P(z) = \frac{1}{3}\sum_{k \in Z} P_k z^k$ is called the two scale matrix symbol of the two scale matrix sequence $\{P_k\}_{k \in Z}$ of Φ.

The orthonormal of Φ and Ψ implies the following perfect reconstruction conditions

$$\sum_{j=0}^{2} |P(w_j z)|^2 = 1, \ |z| = 1 \qquad (4)$$

where $w_j = e^{-ij2\pi/3}$, $j = 0, 1, 2$.

(4) is equivalent to

$$\sum_{j \in Z} P_j \overline{P}^T_{j+3k} = 3\delta_{k,0}, \ k \in Z \qquad (5)$$

In order to construct compactly supported orthogonal scaling functions, we follow Daubechies by considering two scale symbols of the form

$$\begin{cases} P(z) = \left(\dfrac{1 + z + z^2}{3}\right)^m S_n(z) \\ S_n \in \pi_n \text{ with } (1 + z + z^2) + S_n(z) \\ \text{and } S_n(1) = 1 \end{cases} \qquad (6)$$

2 The complex-valued uni-scaling function

This section, we will describe a methods to derive complex valued uni-scaling function from a real valued ones by replacing certain filter roots [4,5]. For $a = 3$, LIAN [2] constructed a symmetric compactly supported orthogoal scaling function $\varphi(x)$ which two scale symbol $P(z)$ has the following form:

$$P(z) = \left(\frac{1 + z + z^2}{3}\right)^m S_n(z),$$

where $S_n(z) = \sum_{j=0}^{n} s_j z^n$ is symmetric polynomial, i.e., $s_j = s_{n-j}$.

Proposition 1 Let $S_n(z) = \sum_{j=0}^{n} s_j z^n$ is symmetric polynomial, i.e., $s_j = s_{n-j}$, then its roots are conjugate reciprocal root pairs and self conjugate reciprocal root pairs.

Hence, $P(z)$ admits the factorization:

$$P(z) = \left(\frac{1 + z + z^2}{3}\right)^m C(z-1)^{L_1}(z+1)^{L_2} \cdot \prod_{i=1}^{L_3}(z - \lambda_i)(z - \lambda_i^{-1}) \prod_{i=1}^{L_4}(z - c_i)(z - \overline{c_i}) \cdot \prod_{i=1}^{L_5}(z - z_i)(z - \overline{z}_i^{-1})(z - \overline{z_i})(z - z_i^{-1}) \qquad (7)$$

where C is some constant, λ_i, λ_i^{-1}, $i = 1, 2, \cdots, L_3$ are $S_n(z)$'s real roots; c_i, $i = 1, 2, \cdots, L_4$ are $S_n(z)$'s roots and $c_i = \overline{c}_i^{-1}$; z_i, $\overline{z}_i^{-1}$, $i = 1, 2, \cdots, L_5$ are conjugate reciprocal root pairs and $z_i \ne \overline{z}_i^{-1}$.

Now we can expect to construct the symmetric two-scale symbol $P_C(z)$ from $P(z)$, and then uni-complex symmetric scaling function is obtained.

Proposition 2 Let $P(z)$ defined in (7) be two scale symbol associated with $\varphi(x)$, construct $P_C(z) = (-1)^{L_1 + L_2}\prod_{i=1}^{L_4} c_i \prod_{i=1}^{L_5} \overline{z}_i z_i^{-1} P(z)$, Then $P_C(z)$ is a symmetric polynomial with complex coefficients and therefore a uni-complex symmetric scaling function with dilation factor 3 associated with $P_C(z)$ can be constructed.

Remark 1 $P_C(z)$ obtained by Proposition 2 is not unique, nor does the corresponding uni-complex scaling function.

Remark 2 A lot of $P_C(z)$ are also constructed by other replacing scheme which different the method of Proposition 2.

3 Multiscaling functions with dilation factor 3

Let φ_R and φ_I be φ_C's real and imaginary parts, respectively, i.e.,

$$\varphi_C(x) = \varphi_R(x) + j\varphi_I(x) \tag{8}$$

and satisfy the following two-scale equation

$$\varphi_C(x) = \sum_{k=0}^{m} p_k \varphi_C(3x - k),\ p_0 p_m \neq 0 \tag{9}$$

where $p_k = \alpha_k + j\beta_k$, $k = 0, 1, \cdots, m$, then

$$P_C(z) = P_R(z) + jP_I(z) \tag{10}$$

Here $P_R(z) = \frac{1}{3}\sum_{k=0}^{m}\alpha_k z^k$, $P_I(z) = \frac{1}{3}\sum_{k=0}^{m}\beta_k z^k$.

Let $\Phi(x) = [\varphi_R(x), \varphi_I(x)]^T$, from (11) and (12), we have

$$\Phi(x) = \sum_{k=0}^{m}\begin{bmatrix}\alpha_k & -\beta_k \\ \beta_k & \alpha_k\end{bmatrix}\Phi(3x - k) \tag{11}$$

The Fourier transformation of (14) is

$$\hat{\Phi}(w) = P(z)\hat{\Phi}(\frac{w}{3}) \tag{12}$$

Here

$$P(z) = \begin{bmatrix}P_R(z) & -P_I(z) \\ P_I(z) & P_R(z)\end{bmatrix}.$$

Define

$$P_k = \begin{bmatrix}\alpha_k & -\beta_k \\ \beta_k & \alpha_k\end{bmatrix},$$

then (11) can rewrite as follows:

$$\Phi(x) = \sum_{k=0}^{m} P_k \Phi(3x - k) \tag{13}$$

Theorem 1 Let $\varphi_C(x)$ be a uni-complex orthogonal scaling function, then the multiscaling functions $\Phi(x)$ associated with $\varphi_C(x)$ is also orthogonal.

Proof Since $\varphi_C(x)$ is orthogonal, i.e.,

$$\langle \varphi_C(\cdot), \varphi_C(\cdot - k)\rangle = \delta_{0,k} \tag{14}$$

Equivalently

$$\begin{cases}\langle \varphi_R(\cdot), \varphi_R(\cdot - k)\rangle - \langle \varphi_I(\cdot), \varphi_I(\cdot - k)\rangle = \delta_{0,k} \\ \langle \varphi_R(\cdot), \varphi_I(\cdot - k)\rangle + \langle \varphi_I(\cdot), \varphi_R(\cdot - k)\rangle = 0\end{cases} \tag{15}$$

From (9) and (14), we have

$$\sum_i p_i \overline{p}_{i+3k} = 3\delta_{0,k},\ k \in Z \tag{16}$$

(16) is equivalent to

$$\begin{cases}\sum_i [\alpha_i\alpha_{i+3k} - \beta_i\beta_{i+3k}] = 3\delta_{0,k} \\ \sum_i [\alpha_i\beta_{i+3k} - \beta_i\alpha_{i+3k}] = 0\end{cases} \tag{17}$$

(17) is also equivalent to

$$\sum_i P_i P_{i+3k}^T = 3\delta_{0,k} I_2,\ k \in Z \tag{18}$$

(18) implies multiscaling function being orthogonal. ▭

Theorem 2 Let $\varphi_C(x) = \varphi_R(x) + j\varphi_I(x)$ be compactly supported orthogonal symmetric uni-complex scaling function, satisfying following equation:

$$\varphi_C(x) = \sum_{k=0}^{m}(\alpha_k + j\beta_k)\varphi_C(3x - k) \tag{19}$$

then $\Phi(x) = [\varphi_R(x), \varphi_I(x)]^T$ be a orthogonal symmetric multiscaling functions and satisfying following equation:

$$\Phi(x) = \sum_{k=0}^{m} P_k \Phi(3x - k) \tag{20}$$

where

$$P_k = \begin{bmatrix}\alpha_k & -\beta_k \\ \beta_k & \alpha_k\end{bmatrix},\ k = 0, 1, \cdots, m \tag{21}$$

4 Construction examples

Example 1 Let $\varphi(x)$ be an orthogonal scaling function with dilation factor 3, the corresponding two scale symbol $P(z)$ satisfy the following equations[2]:

$$P(z) = (\frac{1 + z + z^2}{3})^3 S_{10}(z),$$

where $S_{10}(z)$ is symmetric polynomial of order 10, and its coefficients s_j satisfy $s_j = s_{10-j}$ and $\{s_j\}_{j=0}^{5} = \{0.0146266, -1.0438800, 0.2183709, -0.4972924, -0.6397556, 2.8958607\}$.

We follow the construction scheme and apply Proposition 2 in Section 3 to construct uni-complex scaling function as follows:

$$\varphi_C(x) = \sum_{k=0}^{16} p_k \varphi_C(3x - k),$$

here p_k satisfy $p_k = p_{16-k}$ and $\{p_k\}_{j=0}^{8} = \{(9.658508e-4) - (1.30689e-3)j,\ 0 + 0j,\ 1.152281e-2) - (1.559149e-2)j,\ (-2.029054e-4) + (2.745507e-4)j,\ (-0.0687318) + 9.300081e-2)j,\ -0.4609145 + 0.6236621j,$

0.1268874 - 0.171691j, 0.5354145 - 0.7244679j, 0.6634703 - 0.8977399j}. Then the corresponding orthogonal symmetric multiscaling functions is

$$\Phi(x) = \sum_{k=0}^{16} P_k \Phi(3x - k),$$

where $P_k = P_{17-k}$, $k = 0, 1, \cdots, 16$ and

$$P_k = \begin{bmatrix} \alpha_k & -\beta_k \\ \beta_k & \alpha_k \end{bmatrix}, k = 0, 1, \cdots, 8.$$

Example 2 Let $\varphi(x)$ be an orthogonal scaling function with dilation factor 3, the corresponding two scale symbol $P(z)$ satisfy the following equations[5].

$$P(z) = \left(\frac{1 + z + z^2}{3}\right)^4 S_{12}(z),$$

where $S_{12}(z)$ is symmetric polynomial of order 12, and its coefficients s_j satisfy $s_j = s_{12-j}$ and $\{s_j\}_{j=0}^{6} = \{-0.0357678, 0.0895864, 0.0393211, -0.1848403, 0.7100285, -4.0910423, 7.9454287\}$ similar to example 1, we have

$$\varphi(x) = \sum_{k=0}^{17} p_k \varphi(2x - k),$$

here p_k satisfy $p_k = p_{17-k}$ and $\{p_k\}_{j=0}^{8} = \{(5.335946e-4) + (1.21241e-3)j, (7.979015e-4) + (1.812956e-3)j, (-5.965645e-4) - (1.355487-3)j, (-4.416167e-3) - (1.003421e-2)j, (-1.667377e-2) - (3.788539e-2)j, (1.999552e-2) + (4.543292e-2)j, (3.659597e-2) + (8.315171e-2)j, (5.951397e-2) + 0.135225j, -0.1431015 - 0.3251488j, -0.3118046 - 0.7084684j, -0.49007 - 1.1113515j\}$. Then the corresponding orthogonal symmetric multiscaling functions is

$$\Phi(x) = \sum_{k=0}^{21} P_k \Phi(3x - k),$$

where $P_k = P_{21-k}$, $k = 0, 1, \cdots, 21$ and

$$P_k = \begin{bmatrix} \alpha_k & -\beta_k \\ \beta_k & \alpha_k \end{bmatrix}, k = 0, 1, \cdots, 10.$$

5 Conclusion

In this paper, we proposed a scheme to construct the symmetric compactly supported multiscaling function with dilation factor $a = 3$, which proved a new way to construct desirable property multiwavelet with dilation factor $a = 3$. Similarly, the method can also be used to construt multiscaling functions with dilation factor $a > 3$.

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尺度因子为 3 的多尺度函数的构造

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摘 要: 本文给出尺度因子为 $a = 3$ 的紧支撑正交对称多尺度函数的构造格式. 它首先由尺度因子 $a = 3$的实单一的紧支撑尺度函数构造出尺度因子 $a = 3$ 的单一紧支撑正交对称的复尺度函数, 然后再由构造出的复尺度函数构造二重正交紧支撑多尺度函数, 从而为尺度因子 $a = 3$ 的多小波的构造提供一种新途径. 算例表明构造算法是可行且极易实施.

关键词: 紧支撑; 正交; 尺度函数; 尺度因子

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