Flat Connections on Quantum Bundles and[1] Fractional Statistic in Geometric Quantization*

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Abstract

It is shown that in the geometric quantization formulation, fractional statistics in a quantized system of N indistinguishable particles in two spatial dimensions arise from the nontrivial cohomology of the flat connection on the quantum line bundle as well as from the nontrival homology of the configuration space. The propagator of an nonrelativistic interacting system with fractional statistics is derived.

1. Introduction

The possible existence of fractional statistics in (2+1) dimensions has

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been recognized and discussed for many years [1]. It has been suggested that fractional statistics may be relevant to important physical phenomena such as the fractional quantum Hall effect [2] and high T_c superconductivity [3]. Since the unusual statistical property in two spatial dimensions is closely related to the nontrivial topology, namely the global property, of the configuration space of a system of indistinguishable particles [1,4], and since the geometric quantization formulation [5, 6] is essentially a globalization of canonical quantization, we think it natural and important to discuss fractional statistics in the context of this formulation. This paper is devoted to this subject.

In the framework that permits the existence of Fermi-Dirac statistics, the configuration space of N indistinguishable particles in d spatial dimensions is [4]

$$M^{dN} = (R^{dN} - D)/S_N \qquad (1)$$

where $R^{dN} = (R^d)^{\otimes N}$ is the dN-dimensional Euclidean space, $D = \{(\vec{q}_1 \cdots \vec{q}_N) \mid \vec{q}_i = \vec{q}_j,\ i \neq j = 1 \cdots N\}$ is the set of diagonal points, and S_N is the permutation group. The fundamental group $\pi_1(M^{dN})$ of M^{dN} depends decisively on d. For $d \geqslant 3$ it is S_N and for $d = 2$ it is Artin's braid group B_N (M^{dN} is disconnected when $d = 1$). As is well known, the difference between B_N and S_N is responsible for the radically dissimilar statistical properties of systems in $d \geqslant 3$ and in $d = 2$ spaces. We find that in the geometric equantization formulation the crucial factor giving rise to quantum statistics is the flat part of the connection on the quantum line bundle. When the Hilbert space is chosen to be composed of covariantly constant sections along the vertical polarization, that is, when the Schrödinger position representation is selected, a nontrivial choice of the cohomology class of the flat part of the connection combines with the homology of configuration space in $d = 2$ dimensions to give unusual statistics.

In section 2 the method of [5, 6] is used to give a review of the classical geometric description of an N particle system and to quantize it. In section 3 the relation between statistics and the aforementioned connection on the complex line bundle is examined. In section 4 the statistical phase factor is expressed in terms of a nontrivial solution to the equation for the flat connection, and the BKS kernel is used to derive a path-integral expression for the propagator of a nonrelativistic system. Section 5 is contains our conclusions.

2. Geometric Quantization of a System of N Indistinguishable Particles

Following the method of [5, 6], we first construct a symplectic geometry for a system of N indistinguishable particles, whose configuration space M^{dN} is given by (1) and whose phase space, Γ, is the cotangent bundle over M^{dN}. The symplectic manifold (Γ, ω) is the manifold Γ equipped with a nondegenerate closed form ω (called the symplectic form) on Γ. That is,

$$d\omega = 0, \tag{2}$$

$$\omega(X, \cdot) = 0 \Rightarrow X = 0, \tag{3}$$

where X is a tangent vector field on Γ. Associated with each smooth function f, which may be a classical observable, on Γ, define a tangent vector field X_f called the Hamiltonian vector field by

$$X_f \rfloor \omega = -df, \tag{4}$$

where $\rfloor$ denotes the inner product. Define the Poisson bracket of two functions f and g as

$$\{f, g\} \equiv -\omega(X_f, X_g) = -X_f g = X_g f. \tag{5}$$

The closure of ω ensures the Jacobi identity on the Poisson bracket. Let γ be a map of the interval (0, 1) to M^{dN}, $\gamma: (0, 1) \rightarrow M^{dN}$. If f is the Hamiltonian function, then the equation for the intergral curve $\gamma(t)$ of X_f,

$$\frac{d}{dt}\gamma(t) = X_f \gamma(t) \tag{6}$$

is the canonical Hamiltonian equation of motion.

Because ω is closed, it can be expressed locally as

$$\omega = d\theta. \tag{7}$$

where the canonical 1-form symplectic potential θ is defined only up to a closed 1-form. Also ω and θ may be expressed locally in terms of the canonical local coordinates (q_i^a, p_{ia}), $i = 1 \cdots N$, $a = 1 \cdots d$,

$$\omega = \sum_{i,a} dp_{ia} \wedge dq_i^a, \tag{8}$$

$$\theta = \sum_{i,a} p_{ia} dq_i^a. \tag{9}$$

Being dependent only on ω, classical mechanics is invariant under general canonical transformations which always preserve ω.

The first step in geometric quantization is to associate a linear operator

P_f with each function f on Γ such that P_1 is the identity operator and all P_f satisfy the commutation relation

$$[P_f, P_g] = i\frac{h}{2\pi}P_{\{f,g\}}. \qquad (10)$$

This may be achieved by introducing a complex line bundle L, called the prequantization line bundle, over Γ. Denote by L^x the bundle obtained from L by removing its zero section; this is the U(1) principle fibre bundle over Γ associated with L. On L^x define a connection 1-form α and a curvature 2-form Ω related by

$$\Omega = d\alpha. \qquad (11)$$

Denote by η_c the fundamental vector fild on L^x and by $\eta_c(z)$ the tangent vector of the curve $t \to e^{i2\pi ct}z$, where z is the coordinate on a fibre in L^x. Then α satisfies

$$\eta_c \lrcorner \alpha = c, \qquad (12)$$

where $c \in C$ is a complex number. From (11) and (12),

$$\eta_c \lrcorner \Omega = 0 \quad \mathcal{L}_{\eta_c} d\alpha = 0, \qquad (13)$$

where $\mathcal{L}_{\eta_c}$ is the Lie derivative along η_c. Hence, Ω is the pull-back of a closed 2-form on Γ. Thus the general form of α is

$$\alpha = \delta + \beta, \qquad (14)$$

where β is a pull-back of a 1-form on Γ and, according to (12), δ admits a local expression

$$\delta = dz/2\pi iz. \qquad (15)$$

The operators P_f act on the space of sections of the bundle L as follows. The one-parameter group of canonical transformations ϕ^t_f generated by f has a unique lift to a one-parameter group of connection preserving tranformations of the bundle, which defines the action of ϕ^t_f on the sections. The operator P_f is defined by

$$P_f\lambda = i\frac{h}{2\pi}\frac{d}{dt}(\phi^t_f\lambda)\,|_{t=0} = (-i\frac{h}{2\pi}\nabla_{X_f} + f)\lambda, \quad \nabla_{X_f} = X_f + 2\pi i X_f \lrcorner \beta, \qquad (16)$$

where λ is a section of L and ∇_{X_f} is the covariant derivative along the direction X_f. Condition (10) requires

$$\Omega = -h^{-1}\omega \qquad (17)$$

which, from (7) and (11), implies that in general

$$\beta = -\frac{\theta}{h} - \alpha_0, \qquad (18)$$

where α_0 is an arbitrary closed 1-form,

$$d\alpha_0 = 0. \qquad (19)$$

Since α_0 does not contribute to Ω, we call α_0 the flat part of the connection 1-form.

The prequantization space, or the space of sections of Γ, forms the full representation space of quantum physics. However, this space is obviously too big to be the correct physical quantum Hilbert space, since locally a section admits functions of both $q_i{}^a$ and p_{ia}, which would lead to violation of the uncertainty principle. In order to prevent this from happening it is necessary to reduce by 'polarization' the prequantization space to a suitable subspace.

A polarization F of a symplectic manifold (Γ, ω) is an involutive complex distribution on Γ satisfying

$$\dim F = \frac{1}{2}\dim\Gamma, \qquad \omega\mid_{F\times F} = 0. \tag{20}$$

Given a polarization F, the space of sections of L can be restricted to a subspace of sections that are covariantly constant along $\bar{F}$, the complex conjugate distribution of F. The conjugate $\bar{\psi}$ of such a section ψ must be covariantly constant along F. In general, the Hermitian product $\psi\bar{\psi}'$ is variantly constant along $D = F\cap\bar{F}$ and its integral over Γ diverges unless the leaves of D are compact. This may suggest that it should be natural to integrate $\psi\bar{\psi}'$ over Γ/D, except that there is not a natural measure on Γ/D. One way to circumvent this difficulty is to use half-forms to construct the correct density to be integrated over Γ/D. A completely self-contained description of half-forms is lengthy; interested readers are referred to[5] and [6]. Here, we only give a very simple presentation.

Let F be a polarization on Γ which forms an n-dimensional vector space, $\{X_a\}$ an arbitrary basis of F whose linear transformations form the group GL (n; C), and G a matrix representation of an element in GL (n; C). Denote by $\delta_r(F)$ the set of all functions ν on F with the property

$$\nu\{(XG)_a\} = (\det G)^r \nu\{X_a\}. \tag{21}$$

The elements in $\delta_r(F)$ may be thought of as the rth power of the volume element on F. Every $\delta_r(F)$ is a one-dimensional complex vector space and the vector spaces of $\delta_r(F)$ satisfy

$$\delta_0(F) = C, \qquad \delta_1(F) = \wedge^{\mathbb{C}\,n} F^*,$$
$$\delta_r(F) = (\delta_{-r}(F))^* = \delta_{-r}(F^*), \quad \delta_r(\bar{F}) = \overline{\delta_r(F)},$$
$$\delta_r(F)\otimes\delta_s(F) = \delta_{r+s}(F), \tag{22}$$

where * denotes dual and overline denotes complex conjugation. It can be shown that the bundle over Γ whose sections give the correct quantum description of physical states is L $\otimes\delta_{-1/2}(\overline{F})$. Then the polarized wave functions are locally of the form

$$\psi = sv, \tag{23}$$

where s is a (polarized) section in L satisfying

$$\nabla_X s = 0, \quad \forall X \in \{\overline{F}\}, \tag{24}$$

and v is a section of $\delta_{-1/2}(\overline{F})$ satisfying

$$\Delta_X v = 0, \quad \forall X \in \{\overline{F}\}. \tag{25}$$

The sections of $\delta_{-1/2}(F)$ are called half-forms. The inner product is defined as

$$\langle \psi \mid \psi \rangle = \int_{\Gamma/D} s_m \overline{s_m} \mid \overline{v_m} |^2 | \epsilon_m' |^{1/2} | \epsilon_m |, \tag{26}$$

where the subscript m denotes a point in Γ/D,

$$\epsilon_m = (-1)^{n(n-1)/2} \frac{1}{n!} \omega^n \tag{27}$$

is the Liouville form on Γ and ϵ_m' is the Liouville form on $(F_m + \overline{F_m})/D_m$.

The above description is applicable to any polarization. In this paper we focus on the Schrödinger position picture, ie., we take the vertical polarization in which F is spanned by the linear frame fields$\{\frac{\partial}{\partial p_i^a}$ $i = 1 \cdots N$, $a = 1 \cdots d\}$. Then the polarized wave functions have the form

$$\psi = \hat{\psi}(q_i^a) sv, \tag{28}$$

where s is the unit section. Since $\Gamma/D = M^{dN}$, the inner product can be reduced to

$$\int \psi(q) \overline{\psi'}(q) \mid dq_1^1 \wedge \cdots \wedge dq_N^d \mid. \tag{29}$$

The reduction of the full prequantization representation space to the Hilbert space H of sections of $L \otimes \delta_{-1/2}(\overline{F})$, which is covariantly constant along $\overline{F}$, requires that the definition (16) of operators be modified. Consider a function f on Γ and its Hamiltonian vector field X_f. Denote by ϕ_f^t the one-parameter group of canonical transformations generated by X_f with parameter t; ϕ_f^t is a map on the prequantization representation space. For each $t \in R$, the image of the polarization F of Γ under the derived mapping $T\phi_f^t$ is a polarization $F_t = T\phi_f^t(F)$ of the symplectic (Γ, ω). For each $\psi \in H$ one can define $\phi_f^t\psi$. This gives a map ϕ_f^t: $H \to H_t$ whose target space H_t consists of covariantly constant sections of $L \otimes \delta_{-1/2}(F_t)$ along $\overline{F_t}$. If f is a polarization-preserving function, i.e., if $F_t = F$, then a quantum operator O_f on H can be defined as in (16):

$$O_f(\psi) = i \frac{h}{2\pi} \frac{d}{dt} (\phi_f^t \psi) |_{t=0}, \qquad \psi \in H. \tag{30}$$

On the other hand, if f is rot a polarization preserving function, i.e., if $F_t \neq F$, then (see section 5 in [6]) there exsits a Blattner-Konstant-Sternberg kerrel $K_t : H_t \times H \to C$ and a linear map $U_t : H_t \to H$ such that

$$K_t(\psi, \psi_t) = \langle \psi \mid U_t \psi_t \rangle . \qquad (31)$$

For small $t \in (0, \epsilon)$ define $\phi_t : H \to H$ by

$$\phi_t = U_t \circ \phi_t' . \qquad (32)$$

Then the quantum operator is given by

$$O_f = i \frac{h}{2\pi} \frac{d}{dt} \phi_t \mid_{t=0} \qquad (33)$$

Operators defined by (30) and(33)both satisfy the commutation relation(10)

3. Quantum Connection and Statistics

In the last section it was shown that in a given classical system the symplectic 2-form ω is fixed,but in the quantized system the quantum connection 1-form α is determined only up to a flat connection 1-form α_0. Different choices of α_0 correspond to the same classical limit but may give distinct quantum systems. In this section we will show that, for a system of N indistinguishable particles,a choice of the cohomology class to which α_0 belongs is precisely the choice of the statistics of the quantum system. Often statistics is thought of as just the exchange symmetry of the wave functions of indistinguishable particles. However, as is well known [3], the issue is not just exchange symmetry when M^{dN} is not simply connected. To see this, let $\gamma(t)$ be a curve in M^{dN} as before. Since M^{dN} is the base manifold of the cotangent bundle Γ it is a submanifold of the base manifold of the quantization bundle $\hat{L}$. After having been fixed to the vertical polarization the Hilbert space H may be considered as a bundle over M^{dN} whose connection 1-form α provides a way to compare wave functions at different points in M^{dN}. Denote by $V(M^{dN})$ the tangent vector space of M^{dN}. Then, because the representation space is restricted to the Schrödinger position representation,

$$\omega(X_i, X_j) = 0, \qquad \forall X_i, X_j \in V(M^{dN}) \qquad (34)$$

Therefore, from (17), the curvature Ω vanishes, and the connection 1-form α restricted to M^{dN} is flat, or $d\alpha = 0$ on M^{dN}. This means that α on M^{dN} may be classified into equivalence classes according to the first de Rham cohomology $H^1(M^{dN})$ of M^{dN}.

The parallel transport of the wave function ψ along the curve $\gamma(t)$

on M^{dN} is given by the solution to the equation for the horizontal lift of $\gamma(t)$,

$$\nabla_{X_{\gamma(t)}}\psi(\gamma(t))=0, \qquad (35)$$

where $X_{\gamma(t)}$ is the tangent vector field of $\gamma(t)$. The equation has the formal solution

$$\psi(\gamma(t))=\exp\{-i\int_{\gamma(t)}\alpha\}\psi(\gamma(0)) \qquad (36)$$

Consider the case when $\gamma(t)$ is a closed curve: $\gamma(1)=\gamma(0)$. Although $\gamma(1)$ and $\gamma(0)$ are the same point in M^{dN} the wave functions $\psi(\gamma(1))$, and $\psi(\gamma(0))$ differ by the phase factor

$$x(\gamma)=\exp\{-i\oint_{\gamma}\alpha\}, \qquad (37)$$

which is an element of the holonomy group of connections. Its value is determined by the de Rham cohomology class to which α belongs and the homology class $\{\gamma\}$ to which γ belongs. For a given α (37) gives a one-dimensional representation of the homology group $H_1(M^{dN})$. Since

$$H_1(M^{dN})=Z_2, \qquad d\geqslant 3, \qquad (38)$$

$$H_1(M^{2N})=Z, \qquad (39)$$

we have, for $d\geqslant 3$,

$$\chi(\gamma)=\begin{cases}1, & \forall\alpha\in \text{trivial } H^1(M^{dN})\\ \pm 1, & \forall\alpha\in \text{nontrivial } H^1(M^{dN})\end{cases} \qquad (40)$$

and for $d=2$,

$$\chi(\gamma)=\exp(in_\gamma\Theta), \qquad \Theta\in R,\ n_\gamma\in Z. \qquad (41)$$

One recognizes the χ in (40) to be just the one-dimensional representations of the permutation group S_N: bosonic in the upper case and fermionic in the lower case. In (41) n_γ characterizes the homology class to which γ belongs and the parameter Θ is determined by the choice of the connection 1-form $\alpha\in H_1(M^{2N})$. Different values of Θ implies different statistics.

4. Fractional Statistics in the d = 2 Case

We have seen that the connection 1-form α on the quantization bundle $\hat{L}$ satisfies the equations

$$h^{-1}\omega=-d\alpha, \quad \text{on } \Gamma, \qquad (42)$$

$$d\alpha\,|_{V(M^{dN})}=0 \qquad (43)$$

That is, α is generally determined up to a flat connection 1-form α_0 and is itself flat when the vertical polarization is chosen. In the latter case only the de Rham cohomology class of $H^1(M^{dN})$ to which α belongs matters to the phase factor $\chi(\gamma)$. In what follows we focus on the case $d=2$. Then locally, α admits the expression

$$-\alpha=-\delta+\sum_{i=1}^{N}\sum_{a=i}^{2}p_{ia}dq_i^a+\frac{\Theta}{\pi}\sum_{i<j}^{N}d\phi_{ij}(q), \qquad (44)$$

where ϕ_{ij} is the azimuthal angle of particle i relative to particle j. The last term on the right-hand side of (44) is α_0. This expression obviously obeys (42) and (43) and gives the result for the integral in (37)

$$-\oint_{\gamma}\alpha=\frac{\Theta}{\pi}\sum_{i<j}^{N}n_{ij}\pi, \qquad (45)$$

where $n_{ij}=\oint_{\gamma}d\phi_{ij}/\pi$ is the winding number of the jth particle around the ith particle. Thus the phase factor is

$$\chi(\gamma)=e^{i\Theta\sum_{i<j}^{N}n_{ij}}=e^{i\Theta n_{\gamma}}. \qquad (46)$$

The choice $\Theta=0$ (mod 2π) gives Bose-Einstein satistics, $\Theta=\pi$ (mod 2π) gives Fermi-Dirac statistics, $\Theta=$ raticral number gives fractional statistics and any other choice of real number gives Θ-statistics.

A nontrivial choice of α_0 may also contribute a statistical factor to the propagator. To see this, consider a system of N indistinguishable particles interacting with a Hamiltonian

$$H=\frac{1}{2m}\sum_{i=1}^{N}\sum_{a=1}^{2}(p_{ia})^2+V(q_i^a). \qquad (47)$$

Since H does not preserve the vertical polarization, the evolution from $t=0$ to t generated by H brings the Hilbert space from H to H_t. According to the discussion given in section 2 the method outlined in (31-33) should be used to quantize the system. After a tedious calculation using the technique of BKS kernel (see section 7 in [6] for detail) one obtains for the propagator the expression

$$Z(0,\ q_0;\ t,\ q_t)=\int Dq(\gamma(s))\exp\{ih^{-1}\int_{\gamma(s)}[-\alpha\lrcorner X_H-H]ds\}, \qquad (48)$$

where $\gamma(s)$ are curves mapping the interval $(o,\ t)$ to M^{dN}, $q_0=q(\gamma(0))$, $q_t=q(\gamma(t))$ and X_H is the Hamiltonian vector feld of H. Substituting the connection 1-form α given in (44) into (48) obtains

$$-\alpha\lrcorner X_H-H=\sum_{i=1}^{N}\sum_{a=1}^{2}p_{ia}dq_i^a\lrcorner X_H-H+\frac{\Theta}{\pi}\sum_{i<j}^{N}d\phi_{ij}\lrcorner X_H$$

$$=\mathcal{L}+\frac{\Theta}{\pi}\sum_{i<j}^{N}d\phi_{ij}\lrcorner X_H, \qquad (49)$$

wherer $\mathcal{L}$ is the classical Lagrangian corresponding to H. Denote by $\{\zeta\}$ the set of homotopy classes of γ's and write for any class ζ

$$\chi(\zeta)=\exp\left\{i\frac{\Theta}{\pi}\sum_{i<j}^{N}(\Delta\phi_{ij})\zeta\right\}, \qquad (50)$$

where along any curve γ in the class ζ, the change in ϕ_{ij},

$$(\Delta\phi_{ij})_{\zeta} = \int_{\gamma(s)\in\zeta} d\phi_{ij} \rfloor X_H ds, \quad (51)$$

depends only on ζ. In terms of $\chi(\zeta)$, (48) becomes

$$Z(0,q_0;\ t,q_t) = \sum_{\{\zeta\}} \chi(\zeta) \int_{\gamma\in\zeta} Dq(\gamma)\exp\left\{i\int_0^t L(\gamma)\right\}. \quad (52)$$

This is just equation (1) in [4].

5. Conclusion

We have seen that in the geometric quantization formalism there exist on the quantization bundle different admissible choices of the flat part α_0 of the connection 1-form, to which the classical limit is insensitive. Different statistics of the quantum system are due to the inequivalent de Rham cohomology classes $H^1(M^{dN})$ to which α_0 belongs. In the case of $d=2$, for which the topology of the configuration space of N indistinguishable particles is nontrivial, fractional statistics is a consequence of the existence of nontrivial de Rham cohomology classes on M^{dN}. Here we have only discussed cases in which the Hilbert space is composed of sections of the line bundle. The extension to cases with internal degrees of freedom which give rise to more complicated bundles and Hilbert spaces would of course be extremely interesting.

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