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A Vertex Algebra Structure on the Representation V_Q of Untwisted Affine Lie Algebra $\widetilde{Sl(n,\mathrm{C})}$

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Abstract: The vertex operator structure on the representation V_Q of untwisted affine algebra associated with $\widetilde{Sl(n,\mathrm{C})}$ is studied by using the representation theory of Lie algebra. Moreover, it is proved that V_Q is a vertex operator algebra according to the calculus methods of formal distributions, and then the conformal vector on the vertex algebra V_Q is given.

Key words: vertex operator; vertex algebra; n-th product

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仿射李代数的$\widetilde{Sl(n,\mathrm{C})}$顶点算子表示 V_Q 上的顶点代数结构

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摘 要: 利用李代数表示论研究了仿射李代数$\widetilde{Sl(n,\mathrm{C})}$的顶点算子表示 V_Q 上的顶点代数结构, 并利用形式幂级数的计算方法证明 V_Q 是一个顶点代数, 然后给出了它上面的保角向量.

关键词: 顶点算子; 顶点代数; n 次积

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0 Introduction

The physicists brought forward the concept of vertex operation algebra in studying the theory of field and string. It is important in studying reprensention theory and finite group. Meurmen and Lepowsky sloved the Guss of Mckay-Thompson with this theory. And Borcherds used the vertex operation algebra and Kac-Moody Lie algebra to slove the famous problem of the Monstrous Moonishine Conjecture and won fields award in 1998. Frenkel and Kac [1-2] had constructed the level-one representations of affine Kac-Moody algebras $A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ by means of vertex operators in 1981. In addition, Xu and Jiang [3] have introduced another set of vertex operators in 1990 which are constructed for the level-one representations of the cases $B_n^{(1)}$ and $G_2^{(1)}$. Xu [4] gave the level-one representations of the affine Lie algebras with first kind in 1991.

Using these vertex operators to construct a vertex operator algebra has been an important subject of study. The representations V_Q of Kac-Moody Lie algebra associated to $\widetilde{Sl(2,\mathrm{C})}$ are constructed, which are based on a certain untwisted or twisted vertex operators, and it is proved to be a vertex operator algebra in Ref. [5]. In this paper, we use the vertex operators of affine Lie algebra $A_n^{(1)}$ with first kind to construct vertex operator algebras. But Jacobi identity for the definition of vertex operator algebra is very complicated. We instead it with the axiom of locality to construct vertex

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operator algebra.

1 Untwisted vertex representation V_Q of $A_n^{(1)}$

In this section, we briefly introduce the structure of V_Q and vertex operators $Y(v,z)$ on V_Q.

Let $B(x,y)$ be the Killing form of a finite dimensional complex simple Lie algebra $Sl(n,\mathbf{C})$. Let $\mathscr{h}$ ($\dim(\mathscr{h}) = n$) be a Cartan subalgebra of $Sl(n,\mathbf{C})$ and Δ be the root system. Then $Sl(n,\mathbf{C}) = \mathscr{h} + \sum_{\alpha\in\Delta} g_a, g_a$ is the root subspace decomposition of g by the Cartan subalgebra $\mathscr{h}$. Denote $\Pi = \{\alpha_1,\cdots,\alpha_n\} \subset \mathscr{h}^*$ a simple root system, where $\mathscr{h}^*$ is the dual space of $\mathscr{h}$.

The root lattice

$$L = \{\alpha = \sum_{i=1}^{n} m_i\alpha_i \mid m_1,\cdots,m_n \in \mathbf{Z}\} \subset \mathscr{h}_{\mathrm{R}}^* \quad (1)$$

is an abelian addition group in the real linear space $\mathscr{h}_{\mathrm{R}}^*$. Then $\mathscr{h}_{\mathrm{R}}^*$ has an inner product $(x,y) = c_0 B(x,y), \forall x,y \in \mathscr{h}_{\mathrm{R}}^*$, where c_0 is a positive constant. The group algebra $\mathbf{C}(e^L)$ of L is an abelian associative algebra with the basis $\{e^\alpha \mid \alpha \in L\}$, where $e^0 = 1$ and $e^\alpha e^\beta = e^{\alpha+\beta}$.

Denote $h_i(m) = t^m \otimes \alpha_i, m \in \mathbf{Z}, 1 \leqslant i \leqslant n, X_m(\alpha) = t^m \otimes e_\alpha, \forall \alpha \in \Delta$, where t is a complex parameter. Let S^- be the complex linear space spanned by the basis

$$1, h_i(-m), m \in \mathbf{Z}^+, 1 \leqslant i \leqslant n.$$

Denote $S(S^-)$ be the symmetric tensor algebra over $\mathbf{C}$ generated by S^- with the product $\vee$. Then $S(S^-)$ is a commutative associated algebra with the unit element 1 and has a basis

$$1, h_{i_1}(-m_1) \vee \cdots \vee h_{i_s}(-m_s),$$
$$1 \leqslant i_1 \leqslant \cdots \leqslant i_s \leqslant n, m_1,\cdots,m_s \in \mathbf{Z}^+, s \in \mathbf{Z}^+.$$

Let $V = S(S^-) \otimes C(e^L)$. The formal linear combination of finite or infinite elements of the basis forms a complete space V_Q of V. It is well known that V_Q is an associative algebra with

$$(h_{i_1}(-m_1) \vee \cdots \vee h_{i_s}(-m_s)) \otimes e^\alpha = (1 \otimes e^\alpha) \prod_{k=1}^{s} (h_{i_k}(-m_k \otimes e^0). \quad (2)$$

Hence the representation space V has a basis $1 \otimes e^\beta, (h_{i_1}(-m_1) \vee \cdots \vee h_{i_s}(-m_s)) \otimes e^\beta = (1 \otimes e^\beta) \prod_{k=1}^{s} ((h_{i_k}(-m_k) \otimes e^0)$, where $\beta \in L, 1 \leqslant i_1 \leqslant \cdots \leqslant i_s \leqslant n, m_1,\cdots,m_s \in \mathbf{Z}^+$, $s = 1,2,\cdots$.

For any $u \otimes e^\beta \in V_Q$, the degree of $u \otimes e^\beta$ is defined by

$$\deg(u \otimes e^\beta) = \deg(u) + \frac{1}{2}(\beta,\beta),$$

where $\deg(u)$ is defined by

$$\deg(1) = 0,$$
$$\deg(h_{i_1}(-m_1) \vee \cdots \vee h_{i_s}(-m_2)) = \sum_{k=1}^{s} m_k.$$

Now, we introduce some linear operators acting on V for the definition of the vertex operator representation of the affine Lie algebra, vertex operator algebra and vertex algebra.

(I) Let $D: V_Q \to V_Q$ be a linear operator, which is defined by

$$D(v \otimes e^\beta) = \deg(v \otimes e^\beta)(v \otimes e^\beta). \quad (3)$$

(II) Let $\partial_{h_i(m_i)}, 1 \leqslant i \leqslant n, m_i \in \mathbf{Z}$ be the linear differential operators acting on the linear space $S(S^-)$, which are defined by

$$\partial_{h_i(m_i)}(h_j(-m_j)) = m_i\delta_{m_i,-m_j}(\alpha_i,\alpha_j), m_i,m_j \in \mathbf{Z}^+. \quad (4)$$

Let $\alpha_i(m_i)$ be a linear operator acting on V_Q, which is defined by

$$\begin{cases} \alpha_i(-m_i)(v \otimes e^\beta) = (h_i(-m_i) \vee v) \otimes e^\beta, \text{when } m_i \in \mathbf{Z}^+, \\ \alpha_i(0)(v \otimes e^\beta) = (\alpha_i,\beta)(v \otimes e^\beta), \\ \alpha_i(m_i)(v \otimes e^\beta) = \partial_{h_i(m_i)}(v) \otimes e^\beta, \text{when } m_i \in \mathbf{Z}^+, \end{cases}$$

where $v \in S(S^-), e^\beta \in \mathbf{C}(e^L)$.

Lemma 1

$$[\alpha_i(m_i), \alpha_j(q_j)] = m_i\delta_{m_i,-q_j}(\alpha_i,\alpha_j)\,\mathrm{id}, m_i,q_j \in \mathbf{Z}. \quad (5)$$

Proof This formula has been easily proved [6-7]. □

Let $\alpha = \sum_{i=1}^{n} a_i\alpha_i \in L$. The linear operator $\alpha(m)$ acting on V is defined by

$$\alpha(m) = \sum_{i=1}^{n} a_i\alpha_i(m), \forall m \in \mathbf{Z}.$$

By the induction, we have

Lemma 2 Let

$$\alpha(m) = \sum_{i=1}^{n} a_i\alpha_i(m), \beta(q) = \sum_{j=1}^{n} b_j\alpha_j(q),$$

then

$$[\alpha(m), \beta(q)] = m\delta_{m+q,0}(\alpha,\beta)\,\mathrm{id}. \quad (6)$$

Particularly,

$$\exp(\alpha(m))\exp(\beta(-m)) = \exp(m(\alpha,\beta))\exp(\beta(-m))\exp(\alpha(m)).$$

(III) The mapping $\varepsilon: L \times L \to \{e^{i\theta} \mid 0 \leqslant \theta < 2\pi\}$ is called the ε-mapping, if ε satisfies the following conditions:

(i) $\varepsilon(0,\beta) = \varepsilon(\beta,0) = 1, \forall \beta \in L$;

(ii) $\varepsilon(\alpha,\beta) = (-1)^{(\alpha,\beta)}\varepsilon(\beta,\alpha), \forall \alpha,\beta \in L$;

(iii) $\varepsilon(\beta+\gamma,\alpha)\varepsilon(\beta,\gamma) = \varepsilon(\beta,\alpha+\gamma)\varepsilon(\gamma,\alpha), \forall \alpha, \beta,\gamma \in L$.

(iv) Let $x(\alpha,z): \mathbf{C}(e^L) \to \mathbf{C}(e^L)$ be a linear operator, which is defined by

$$x(\alpha,z)(v \otimes e^\gamma) = \varepsilon(\alpha,\gamma)z^{(\alpha,\gamma)}(v \otimes e^{\alpha+\gamma}). \quad (7)$$

Let $X(\alpha,z) = \sum_{m=-\infty}^{\infty} X_m(\alpha) z^{-m}$ be the Laurent series of z. Then

$$X(\alpha,z)(v\otimes e^\gamma) = E^+(\alpha,z)E^-(\alpha,z)x(\alpha,z)(v\otimes e^\gamma) = \exp(\sum_{m=1}^{\infty}\frac{z^m}{m}\alpha(-m))\exp(\sum_{m=1}^{\infty}\frac{-z^{-m}}{m}\alpha(m))\cdot \varepsilon(\alpha,\gamma)z^{(\alpha,\gamma)}(v\otimes e^{\alpha+\gamma}).$$

Theorem 1 The vertex operator representation (ρ,V) of affine Lie algebras with the first kind can be defined on the generators by

$$\begin{cases}\rho(c) = \mathrm{id},\\ \rho(d) = D,\\ \rho(t^m\otimes\alpha_i) = \alpha_i(m), 1\leqslant i\leqslant n, m\in\mathbf{Z},\\ \rho(t^m\otimes e_\alpha) = X_m(\alpha), \alpha\in\Delta, m\in\mathbf{Z}.\end{cases} \tag{8}$$

This theorem is proved in Ref. [8]. In this case, the multiplication table of the affine Lie algebra is

$$[\mathrm{id},D] = [\mathrm{id},\alpha_i(m)] = [\mathrm{id},X_m(\alpha)] = 0,$$
$$[D,\alpha_i(m)] = m\alpha_i(m),$$
$$[D,X_m(\alpha)] = mX_m(\alpha),$$
$$[\alpha_i(m),\alpha_j(k)] = m\delta_{m,-k}(\alpha_i,\alpha_j)\,\mathrm{id},$$
$$[\alpha_i(m),X_k(\alpha)] = (\alpha,\alpha_i)X_{m+k}(\alpha),$$
$$[X_m(\alpha),X_k(-\alpha)] = \varepsilon(\alpha,-\alpha)(\alpha(m+k) + m\delta_{m,-k}\mathrm{id}),$$
$$[X_m(\alpha),X_k(\beta)] = 0, \alpha,\beta\in\Delta, \alpha+\beta\notin\Delta\cup\{0\},$$
$$[X_m(\alpha),X_k(\beta)] = \varepsilon(\alpha,\beta)X_{m+k}(\alpha+\beta), \quad \alpha,\beta,\alpha+\beta\in\Delta.$$

2 The structure of vertex algebra on V_Q

Definition 1 A complex linear space V is called a vertex algebra, if there exist a set of linear operators (every linear operator is called a field) for v:

$$Y(v,z) = \sum_{m\in\mathbf{Z}} v_{(m)}z^{-m-1} \in \mathrm{End}V[[z,z^{-1}]] \tag{9}$$

such that given any $v,w\in V$, there is a positive integer $m_0 = m_0(v,w)$ such that $v_{(m)}(w) = 0, \forall m > m_0$. And there is a fixed vector $|0\rangle \in V$, which is called by the vacuum vector, such that

(i) (vacuum)

$Y(|0\rangle,z) = \mathrm{id}_V, Y(v,z)|0\rangle|_{z=0} = v.$

(ii) (translation convariance)

$T\in\mathrm{End}(V)$ is defined by $T(v) = v_{(-2)}|0\rangle, \forall v\in V$. T is called the infinitesimal translation operator, if T is a derivation on V and satisfies the condition: $\mathrm{ad}(T) = \partial_z$ acting on any linear operator $Y(u\otimes e^\gamma,z)$.

(iii) (locality)

There exists a positive integer N such that

$(z-w)^N[Y(u,z),Y(v,w)] = 0, \forall u,v\in V.$

This definitions is the same as that in Refs. [9-10].

Define the map $Y(\cdot,z): V_Q\to(\mathrm{End}V_Q)[[z,z^{-1}]], v\to \sum_{n\in\mathbf{Z}} v_n z^{-n-1}$ by the following way:

$$Y(1,z) = \mathrm{id}_{V_Q};$$
$$Y(h_i(-1)\otimes 1,z) = H_i(z) = \sum_{m\in\mathbf{Z}} H_i(m)z^{-m-1};$$
$$Y(h_i(-m)\otimes 1,z) = \partial^{(m-1)}H_i(z);$$
$$Y(1\otimes e^\alpha) = X(\alpha,z) = E^+(\alpha,z)E^-(\alpha,z)x(\alpha,z);$$
$$Y(h_{i_1}(-m_1)\vee\cdots\vee h_{i_s}(-m_s)\otimes 1,z) = :\partial^{(m_1-1)}H_{i_1}(z)\cdots\partial^{(m_s-1)}H_{i_s}(z):;$$
$$Y(h_{i_1}(-m_1)V\cdots Vh_{i_s}(-m_s)\otimes e^\beta,z) = :\partial^{(m_1-1)}H_{i_1}(z)\cdots\partial^{(m_s-1)}H_{i_s}(z)Y(1\otimes e^\beta,z):,$$

where $:\cdot:$ is the normal order of fields or operators, $\partial^{(m)} = \frac{\partial^m}{m!}$ is a differential operator, and the vacuum vector $|0\rangle = 1\otimes e^0$.

In the following, we shall check that $(V,Y(v\otimes e^\alpha,z))$ is a vertex algebra, i.e., it satisfies the three axioms of the vertex algebra.

2.1 The vacuum axiom

(1) $Y(1\otimes 1,z) = \mathrm{id}_V$;

(2) $Y(h_i(-m)\otimes 1,z)1\otimes 1|_{z=0} = h_i(-m)\otimes 1$;

(3) $Y(v\otimes e^\alpha,z)1\otimes 1|_{z=0} = v\otimes e^\alpha$.

This formula can be easily proved by Theorem 1.

2.2 The translation covariance axiom

Definition 2 Let

$$v = \prod_{k=1}^{s} h_{i_k}(-m_k), v_k = \prod_{1\leqslant j\leqslant s, j\neq k} h_{i_j}(-m_j), m_i\in\mathbf{Z}^+.$$

Then

$$T(v\otimes e^\gamma) = \gamma(-1)(v\otimes e^\gamma) + \sum_{k=1}^{s} m_k h_{i_k}(-(1+m_k))(v_k\otimes e^\gamma).$$

T is a derivation which acts on V_Q. Particularly,

$$T(1\otimes e^\gamma) = \gamma(-1)(1\otimes e^\gamma),$$
$$T(h_i(-m)\otimes 1) = mh_i(-m-1)\otimes 1.$$

Lemma 3 Given $i = 1,2,\cdots,n, m\in\mathbf{Z}_+, m>0, \alpha\in L$, then

(i) $\mathrm{ad}(T)\alpha(-m) = m\alpha(-(1+m))$;

(ii) $\mathrm{ad}(T)\alpha(1)(v\otimes e^{\alpha+\gamma}) = -(\alpha,\alpha+\gamma)(v\otimes e^{\alpha+\gamma})$;

(iii) $\mathrm{ad}(T)\alpha(m) = -m\alpha(m-1)$;

(iv) $\mathrm{ad}(T)x(\alpha,z) = \alpha(-1)x(\alpha,z)$.

(v) $\mathrm{ad}(T)E^+(\alpha,z) = E^+(\alpha,z)\sum_{m=2}^{\infty} z^{m-1}\alpha(-m)$;

(vi) $\mathrm{ad}(T)E^-(\alpha,z) = (\sum_{m=1}^{\infty} z^{-m-1}\alpha(m) + z^{-1}(\alpha,\alpha+\gamma))E^-(\alpha,z)$;

$$E^-(\alpha,z)[T,x(\alpha,z)](v\otimes e^\gamma) = (\alpha(-1) - z^{-1}(\alpha,\alpha))E^-(\alpha,z)x(\alpha,z)(v\otimes e^\gamma).$$

Proof The formulas (i-iii,v) is easily proved [11]. Now, we only check the formulas (iv).

$$\mathrm{ad}(T)\,E^+(\alpha,z) = \mathrm{ad}(T)\prod_{m=1}^{\infty}\exp\Big(\frac{1}{m}z^m\alpha(-m)\Big) =$$
$$\sum_{j=1}^{\infty}\prod_{m=1}^{j-1}\exp\Big(\frac{1}{m}z^m\alpha(-m)\,\mathrm{ad}(T)\Big)\cdot$$
$$\exp\Big(\frac{1}{j}z^j\alpha(-j)\Big)\prod_{m=j+1}^{\infty}\exp\Big(\frac{1}{m}z^m\alpha(-m)\Big) =$$
$$E^+(\alpha,z)\sum_{j=1}^{\infty}z^j\alpha(-(1+j)).$$

Hence (iv) holds. □

Lemma 4 The infinitesimal translation operator T satisfies the translation covariance axioms

$$[T,Y(v\otimes e^\gamma)] = \partial_z Y(v\otimes e^\gamma). \qquad (10)$$

Proof Since ∂_z is a differential operator acting on $Y(u\otimes e^\alpha)$ about z, then

$$\partial_z(Y(1\otimes e^\alpha,z))(v\otimes e^\gamma) =$$
$$\partial_z(E^+(\alpha,z)E^-(\alpha,z)x(\alpha,z))(v\otimes e^\gamma) =$$
$$\partial_z(E^+(\alpha,z))E^-(\alpha,z)x(\alpha,z) +$$
$$E^+(\alpha,z)\partial_z(E^-(\alpha,z))x(\alpha,z) +$$
$$E^+(\alpha,z)E^-(\alpha,z)\partial_z(x(\alpha,z))(v\otimes e^\gamma) =$$
$$E^+(\alpha,z)\Big(\sum_{m=1}^{\infty}z^{m-1}\alpha(-m) + \sum_{m=1}^{\infty}z^{-m-1}\alpha(m) +$$
$$z^{-1}(\alpha,\gamma)\Big)E^-(\alpha,\sqrt{z})(\partial_z\tilde{x}(\alpha,\sqrt{z}))(v\otimes e^\gamma).$$

By the formulas (iv), (v) and (vi) of Lemma 3, we have

$$[T,Y(1\otimes e^\alpha,z)](v\otimes e^\gamma) =$$
$$[T,E^+(\alpha,z)E^-(\alpha,z)x(\alpha,z)](v\otimes e^\gamma) =$$
$$([T,E^+(\alpha,z)]E^-(\alpha,z)x(\alpha,z) +$$
$$E^+(\alpha,z)[T,E^-(\alpha,z)]x(\alpha,z) +$$
$$E^+(\alpha,z)E^-(\alpha,z)[T,x(\alpha,z)])(v\otimes e^\gamma) =$$
$$E^+(\alpha,z)\Big(\sum_{m=1}^{\infty}z^{m-1}\alpha(-m) + \sum_{m=1}^{\infty}z^{-m-1}\alpha(m) +$$
$$z^{-1}(\alpha,\gamma)\Big)E^-(\alpha,z)x(\alpha,z)(v\otimes e^\gamma).$$

Therefore

$$[T,Y(1\otimes e^\alpha,z)] = \partial_z(Y(1\otimes e^\alpha,z)).$$

It is easy to see that

$$[T,Y(h_i(-m)\otimes e^0,z)] = \partial_z(Y(h_i(-m)\otimes e^0,z))$$

by the formulas (ii), (iv) and (vi) of Lemma 3. By the induction, we have

$$[T,Y(v\otimes e^0,z)] = \partial_z(Y(v\otimes e^0,z)),$$

and

$$[T,Y(u\otimes e^\alpha,z)] = \partial_z(Y(u\otimes e^\alpha,z)),$$

where $u = \prod_{k=1}^{s}h_{i_k}(-m_k)$. Then

$$\mathrm{ad}(T)Y(h_{i_0}(-m_0\vee u\otimes e^\alpha,z) =$$
$$\mathrm{ad}(T):\partial^{(m_0-1)}H_{i_0}(z)Y(u\otimes e^\alpha): =$$
$$\mathrm{ad}(T)(\partial^{(m_0-1)}H_{i_0}(z)_+Y(u\otimes e^\alpha)) +$$
$$\mathrm{ad}(T)(Y(u\otimes e^\alpha))\partial^{(m_0-1)}H_{i_0}(z)_-$$
$$(\mathrm{ad}(T)\partial^{(m_0-1)}H_{i_0}(z)_+)Y(u\otimes e^\alpha) +$$
$$\partial^{(m_0-1)}H_{i_0}(z)_+[\mathrm{ad}(T)Y(u\otimes e^\alpha)] +$$
$$(\mathrm{ad}(T)Y(u\otimes e^\alpha))\partial^{(m_0-1)}H_{i_0}(z)_- +$$
$$Y(u\otimes e^\alpha)(\mathrm{ad}(T)\partial^{(m_0-1)}H_{i_0}(z)_-) =$$
$$\partial_z(Y(h_{i_0}(-m_0)\vee u\otimes e^\alpha)) +$$
$$((\mathrm{ad}(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_+)Y(u\otimes e^\alpha) +$$
$$Y(u\otimes e^\alpha)(\mathrm{ad}(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_-).$$

Hence we need to prove that

$$((\mathrm{ad}(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_+)Y(u\otimes e^\alpha) +$$
$$Y(u\otimes e^\alpha)((\mathrm{ad}(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_-) = 0.$$

From the formulas (i) and (iii), it is easy to prove that

$$((\mathrm{ad}(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_+) = 0,$$
$$((\mathrm{ad}(T)-\partial_z)\partial^{(m_0-1)}H_{i_0}(z)_-) = 0.$$

Hence (10) holds. □

2.3 The locality axiom

In this subsection, we check the locality axiom. Namely, we will prove the following formula holds.

$$(z-w)^N[Y(u\otimes e^\alpha,z),Y(v\otimes e^\beta)] = 0,$$
$$\forall\, u\otimes e^\alpha, v\otimes e^\beta \in V_Q, z,w\in\mathbf{C}.$$

Lemma 5　$Y(h_i(-1)\otimes 1,z)$ and $Y(h_j(-1)\otimes 1,w)$ are local, i. e.,

$$(z-w)^N[Y(h_i(-1)\otimes 1,z),Y(h_j(-1)\otimes 1,w)] = 0.$$

Proof　Notice that

$$[Y(h_i(-1)\otimes 1,Y(h_j(-1)\otimes 1,w)] =$$
$$\sum_{m\in\mathbf{Z}}\sum_{n\in\mathbf{Z}}[H_i(m),H_j(n)]z^{-m-1}w^{-n-1} =$$
$$(\alpha_i,\beta_j)\sum_{m\in\mathbf{Z}}mz^{-m-1}w^{-m-1} = (\alpha_i,\beta_j)\partial_w\delta(z-w).$$

By the formula (iv) of Lemma 3, it is easy to prove

$$(z-w)^2[Y(h_i(-1)\otimes 1,Y(h_j(-1)\otimes 1,w)] =$$
$$(\alpha_i,\beta_j)(z-w)^2\partial_w\delta(z-w) = 0.$$

Lemma 6[12]　$Y(h_i(-m_i)\otimes 1,z)$ and $Y(h_j(-n_j)\otimes 1,z)$ are local, i. e., $(z-w)^{m_i+n_j}[Y(h_i(-m_i)\otimes 1,z),Y(h_j(-n_j)\otimes 1,w)] = 0$.

Proof Since

$$Y(h_i(-m_i)\otimes 1,z)\cdot Y(h_j(-n_j)\otimes 1,w) =$$
$$Y(h_j(-n_j)\otimes 1,w)\cdot Y(h_i(-m_i)\otimes 1,z) +$$
$$(\alpha_i,\alpha_j)\sum_{m=0}^{\infty}z^m w^{-m-m_i-n_j}\cdot$$
$$\frac{(m+m_i-1)!(m+m_i+n_j-1)!}{m!(m_i-1)!(m-m_i)!(n_j-1)!}\cdot$$
$$(-m-m_i)a_{n_j}(n_j) +$$
$$(\alpha_i,\alpha_j)\sum_{n=0}^{\infty}z^{-n-m_i-n_j}w^n\cdot$$
$$\frac{(n+m_i+n_j-1)!(n+n_j-1)!}{(n+n_j)!(m_i-1)!n!(n_j-1)!}\cdot$$
$$(n+n_j)a_{m_i}(m_i),$$

then

$$Y(h_i(-m_i)\otimes 1,z)\cdot Y(h_j(-n_j)\otimes 1,w) -$$
$$Y(h_j(-n_j)\otimes 1,w)\cdot Y(h_i(-m_i)\otimes 1,z) =$$
$$(\alpha_i,\alpha_j)\Big(\sum_{n=0}^{\infty}z^{-n-m_i-n_j}w^n\cdot$$

$$\frac{(n+m_i+n_j-1)!}{(m_i-1)!n!(n_j-1)!}a_{m_i}(m_i) - \sum_{m=0}^{\infty} z^m w^{-m-m_i-n_j}\frac{(m+m_i+n_j-1)!}{(n_j-1)!m!(m_i-1)!}a_{n_j}(n_j)) = (\alpha_i,\alpha_j)\frac{(m_i+n_j-1)!}{(m_i-1)!(n_j-1)!}\cdot$$

$$(\sum_{n=0}^{\infty} z^{-n-m_i-n_j}w^n\frac{(n+m_i+n_j-1)!}{n!(m_i+n_j-1)!}a_{m_i}(m_i) - \sum_{m=0}^{\infty} z^m w^{-m-m_i-n_j}\frac{(m+m_i+n_j-1)!}{m!(m_i+n_j-1)!}a_{n_j}(n_j)).$$

By the formula (iv) of Lemma 3, we have $(z-w)^{m_i+n_j}[Y(h_i(-m_i)\otimes 1,z),Y(h_j(-n_j)\otimes 1,w)] = 0$.

Lemma 7 $Y(h_i\otimes 1,z)$ and $Y(1\otimes e^{\alpha},w)$ are local, i.e., $(z-w)^N[Y(h_i\otimes 1,z),Y(1\otimes e^{\alpha},w)] = 0$.

Proof Since

$$Y(h_i\otimes 1,z) = H_i(z) = \sum_{m\in\mathbf{Z}} H_i(m)z^{-m-1},$$

then

$$[H_i(m),Y(1\otimes e^{\alpha},w)] = w^m(\alpha_i,\alpha)Y(1\otimes e^{\alpha},w),$$

and

$$[Y(h_i\otimes 1,z),Y(1\otimes e^{\alpha},w)] = \delta(z-w)Y(1\otimes e^{\alpha},w).$$

Therefore, $(z-w)[Y(h_i\otimes 1,z),Y(1\otimes e^{\alpha},w)] = 0$.

Lemma 8 $Y(1\otimes e^{\alpha},z)$ and $Y(1\otimes e^{\beta},w)$ are local, i.e., $(z-w)^N[Y(1\otimes e^{\alpha},z),Y(1\otimes e^{\beta},w)] = 0$.

Proof Since $Y(1\otimes e^{\alpha},z) = E^+(\alpha,z)E^-(\alpha,z)x(\alpha,z)$, we have

$$\begin{aligned}Y(1\otimes e^{\alpha},z)Y(1\otimes e^{\beta},w) = {}& x(\alpha,z)x(\beta,w)E^+(\alpha,z)E^-(\alpha,z)\cdot\\ & E^+(\beta,z)E^-(\beta,z) = \\ & z^{-(\alpha,\beta)}(z-w)^{(\alpha,\beta)}x(\alpha,z)x(\beta,w)\cdot\\ & E^+(\alpha,z)E^+(\beta,z)E)\cdot\\ & E^+(\beta,z)E^-(\alpha,z)E^-(\beta,z).\end{aligned}$$

By the same way, we get

$$\begin{aligned}Y(1\otimes e^{\beta},z)Y(1\otimes e^{\alpha},w) = {}& x(\beta,w)x(\alpha,z)E^+(\beta,w)E^-(\beta,w)E^+(\alpha,z)\cdot\\ & E^-(\alpha,z) = \\ & w^{-(\alpha,\beta)}(w-z)^{(\alpha,\beta)}x(\beta,w)x(\alpha,z)E^+(\alpha,z)\cdot\\ & E^+(\beta,z)E^-(\alpha,z)E^-(\beta,z).\end{aligned}$$

From the definition of mapping ε [13], we have

$$z^{-(\alpha,\beta)}(z-w)^{(\alpha,\beta)}x(\alpha,z)x(\beta,w) = w^{-(\alpha,\beta)}(w-z)^{(\alpha,\beta)}x(\beta,w)x(\alpha,z),$$

i.e., $(z-w)[Y(1\otimes e^{\alpha},z),Y(1\otimes e^{\beta},w)] = 0$.

Lemma 9 [13] If $a(z)$, $b(z)$ and $c(z)$ are pairwise mutually local fields, then $a(z)_{(n)}b(z)$ and $c(z)$ are mutually local fields for all $n\in\mathbf{Z}$ (resp. $n\in\mathbf{Z}$). In particular $:z(z)b(z):$ and $:c(z):$ are mutually local fields provided that $a(z)$, $b(z)$ and $c(z)$ are.

Lemma 10 For any $u\otimes e^{\alpha}, v\otimes e^{\beta}\in V_Q$, there exists a nonegative integer N which satisties

$$(z-w)^N[Y(u\otimes e^{\alpha},z),Y(v\otimes e^{\beta},w)] = 0.$$

Proof From the Lemmas 5-8, we know that $Y(h_i(-1)\otimes 1,z)$, $Y(h_j(-m)\otimes 1)$ and $Y(1\otimes e^{\alpha})$ are pairwise mutually local fields. Then by the Lemma 9, we can easily prove Lemma 10.

From the fact that we have checked that $(V,Y(v\otimes e^{\alpha},z))$ satisfies the three axioms. It follows

Theorem 2 $(V,Y(v\otimes e^{\alpha},z))$ is a vertex algebra.

3 The conformal vector of the vertex algebra

Definition 3 A conformal vector of a vertex algebra V is an even vector v such that the corresponding field

$$Y(v,z) = \sum_{n\in\mathbf{Z}} L_n z^{(-n-2)}$$

is a Virasoro field with cetral charge c which has the following properties:

(a) $L_{-1} = T$,

(b) L_0 is diagonalizable on V.

The number is called the central charge of v.

Let $A = ((\alpha_i,\alpha_j))^{-1} = (a_{ij})$, then A is a symmetric matrix. Let

$$v = \frac{1}{2}\sum_{k=1}^{n} a_{jk}H_j(-1)H_k(-1)\mid 0\rangle,$$

then

$$Y(v,z) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk} :H_j(z)H_k(z): = \sum_{n\in\mathbf{Z}} L_n z^{(-n-2)}.$$

So we have

$$\begin{aligned}L_0 = {}& \frac{1}{2}\sum_{j,k}^{n} a_{jk}H_j(0)H_k(0) + \\ & \frac{1}{2}\sum_{j,k}^{n}\sum_{n>0} a_{jk}(H_j(-n)H_k(n) + \\ & H_j(n)H_k(-n));\\ L_{-1} = {}& \frac{1}{2}\sum_{j,k}^{n}\sum_{n\geqslant 0} a_{jk}(H_j(-n-1)H_k(n) + \\ & H_k(-n-1)H_j(n));\\ L_m = {}& \frac{1}{2}\sum_{j,k}^{n}\sum_{n\in\mathbf{Z}} a_{jk}H_j(n)H_k(m-n), m\neq 0.\end{aligned}$$

In the following, we will prove that L_m satisfies the above properties (a) and (b).

By the define of L_0, we have

$$L_0(1\otimes e^{\beta}) = \frac{1}{2}\sum_{j,k}^{n} a_{jk}H_j(0)H_k(0)(1\otimes e^{\beta}) = \frac{1}{2}(\beta,\beta)(1\otimes e^{\beta}), \tag{11}$$

$$\begin{aligned}L_0(h_i(-m)\otimes 1) = {}& \frac{1}{2}\sum_{j,k}^{n}\sum_{n>0} a_{jk}(H_j(-n)H_k(n) + \\ & H_j(n)H_k(-n))h_i(-m)\otimes 1 = \\ & \frac{1}{2}\sum_{j,k}^{n} a_{jk}(m(\alpha_k,\alpha_i)h_j(-m)\otimes 1 + \end{aligned}$$

$$m(\alpha_j,\alpha_i)h_k(-m)\otimes 1) = mh_i(-m)\otimes 1. \quad (12)$$

Then by the inductive method, so

$$L_0(h_1(-m_1)\vee\cdots\vee h_s(-m_s)\otimes e^{\beta}) = (m_1+\cdots+m_s+\frac{1}{2}(\beta,\beta))\cdot (h_1(-m_1)\vee\cdots\vee h_s(-m_s)\otimes e^{\beta}).$$

Thus L_0 satisfies the above properties (a) i. e. L_0 is diagonalizable on V.

By the define of L_{-1}, we have

$$L_{-1}(1\otimes e^{\alpha}) = \alpha(-1)(1\otimes e^{\alpha}),$$

and

$$L_{-1}(h(-m)\otimes 1) = m(h(-m)\otimes 1).$$

By the define of T, we prove that L_{-1} satisfies the above properties (b) i. e. $L_{-1} = T$.

The field $Y(v,z) = \sum_{n\in\mathbf{Z}} L_n z^{(-n-2)}$ is a Virasoro field and $(V,Y(v\otimes e^{\alpha},z))$ is a conformal vertex algebra.

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