DOI: 10.3969/j. issn. 1003-0972. 2012. 03. 002

On Weakly-Abelian Rings

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Abstract: Weakly-abelian rings were investigated. A series of new characterizations and properties on this kind of rings were given and some known results were generalized. The relations between weakly-abelian rings and other classes of rings were discussed.

Key words: weakly-abelian rings; abelian rings; idempotent elements; Jacobson radicals

CLC number: O153.3 **Article character**: A **Article ID**: 1003-0972(2012)03-0285-04

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摘　要: 对弱 Abel 环进行了研究,得到了一系列新的性质和刻划,推广了已知的结果,并讨论了弱 Abel 环与其他环类之间的关系.

关键词: 弱 Abel 环; Abel 环; 幂等元; Jacobson 根

中图分类号: O153.3 **文献标志码**: A **文章编号**: 1003-0972(2012)03-0285-04

0 Introduction

Throughout this paper, all the rings are associative rings with an identity. Let $E(R)$, $C(R)$, $N(R)$ and $J(R)$ denote the set of all idempotents, the center of R, the set of all nil element of R and the Jacobson radical of R, respectively. For any subset X of R, $l(X)$ and $r(X)$ denote the left annihilators and right annihilators of X respectively. We call a ring R an abelian ring if $E(R) \subset C(R)$. It is well known that R is abelian if and only if $eR(1-e)=0$ for each $e \in E(R)$. Many authors have generalized abelian rings to lager classes of rings. Yu[1] induced quasi-duo ring and Chen[2] induced semiabelian ring. A ring R is called left(right) quasi-duo ring, if every maximal left(right) ideals are ideals, and quasi-duo ring is the ring which is both left and right quasi-duo ring. A ring R is called semiabelian if $eR(1-e)=0$ or $(1-e)Re=0$ for each $e \in E(R)$. Abelian rings are quasi-duo rings and semiabelian rings, while the converse is not true. According to Ref. [3], a ring R is said to be weakly-abelian if $eR(1-e) \subset J(R)$ for each $e \in E(R)$, it is also proved that abelian ring, quasi-duo rings and semiabelian rings are weakly-abelian, and gave examples to show that the converse is not held.

Since weakly-abelian rings contain a larger class of rings having extensive properties, we continue the study of weakly-abelian rings in this paper. We introduce the subset $A_J(R)=\{a \mid ea \equiv a(1-e) \pmod{J(R)} \text{ for some } e \in E(R)\}$ of R, with which to give some new characterizations of weakly-abelian rings. Many known results can be obtained as corollaries.

Received date: 2011-11-22; **Revised date**: 2012-02-24; * **. Corresponding author**, E-mail: shuhunwly2006@163.com

Foundation item: The Creative Seed of Hangzhou Normal University

Biography: WAN Ling-yu(1987-), male, born in Langfang, Hebei Province, Master, engaged in non-commutative ring theory.

1 Main Results

Definition 1 A ring R is called weakly-abelian if $eR(1-e) \subset J(R)$ for every $e \in E(R)$.

Let $A_J(R)$ be the subset $\{a \mid ea \equiv a(1-e) \pmod{J(R)}, \exists e \in E(R)\}$ in R.

Theorem 1 The following statements are equivalent for a ring R:

(1) R is weakly-abelian;

(2) $A_J(R) \subset J(R)$;

(3) For each $a \in R$, $ea = a(1-e)$ for some $e \in E(R)$ implies $a \in J(R)$.

Proof (1)$\Rightarrow$(2). For each $a \in A_J(R)$, $ea = e(ea) \equiv ea(1-e) \equiv 0 \pmod{J(R)}$ since R is weakly-abelian. Similarly,

$$(1-e)a \equiv 0 \pmod{J(R)}.$$

Thus, $a = ea + (1-e)a \in J(R)$.

(2)$\Rightarrow$(3). For each $a \in R$, $ea = a(1-e)$ for some $e \in E(R)$, $a \in A_J(R) \subset J(R)$.

(3)$\Rightarrow$(1). Let $a = er(1-e)$, then $ea = a(1-e)$, thus $a \in J(R)$. Therefore R is weakly-abelian. □

Theorem 2 The following statements are equivalent for a ring R:

(1) R is weakly-abelian;

(2) $ae \in J(R)$ implies $ea \in J(R)$ for each $e \in E(R)$, $a \in A_J(R)$;

(3) $ea \in J(R)$ implies $ae \in J(R)$ for each $e \in E(R)$, $a \in A_J(R)$;

(4) $aRe \subset J(R)$ for each $e \in E(R)$, $a \in A_J(R)$;

(5) $eRa \subset J(R)$ for each $e \in E(R)$, $a \in A_J(R)$;

(6) $ae \in J(R)$ implies $eaRe \subset J(R)$ for each $e \in E(R)$, $a \in A_J(R)$;

(7) $ea \in J(R)$ implies $eRae \subset J(R)$ for each $e \in E(R)$, $a \in A_J(R)$;

(8) For each $e \in E(R)$, $re \in J(R)$ implies $erA_J(R)e \subset J(R)$;

(9) For each $e \in E(R)$, $er \in J(R)$ implies $eA_J(R)re \subset J(R)$.

Proof (1)$\Rightarrow$(2), (3), (4), (5), (6), (7), (8), (9) is clear by Theorem 1. It is suffice to show (2), (4), (6), (8)$\Rightarrow$(1), and we can get (3), (5), (7), (9)$\Rightarrow$(1) similarly.

(2)$\Rightarrow$(1). For each $e \in E(R)$, $r \in R$, let

$$a = er(1-e) \in A_J(R), 0 = ae \in J(R),$$

then $ea = er(1-e) \in J(R)$. Thus R is weakly-abelian.

(4)$\Rightarrow$(1). For each $a \in A_J(R)$, ae, $a(1-e) \in J(R)$. Then $a = ae + a(1-e) \in J(R)$.

(6)$\Rightarrow$(1). For each $e \in E(R)$, $r \in R$, let $a = er(1-e) \in A_J(R)$. $0 = ae \in J(R)$, then $er(1-e)Re \subset J(R)$, thus $eR(1-e)Re \subset J(R)$. By [1, Theorem 1.2.1], R is weakly-ablian.

(8)$\Rightarrow$(1). For each $e \in E(R)$, $r \in R$, $er(1-e)e = 0 \in J(R)$, and for each $r_1 \in R$,

$$(1-e)r_1e \in A_J(R).$$

Then, $er(1-e)r_1e \in J(R)$, thus

$$eR(1-e)Re \subset J(R).$$

By [1, Theorem 1.2.1], R is weakly-ablian. □

Let $N_J(R) = \{r \mid r^2 \in J(R)\}$. It is clear that $N(R) \subset N_J(R)$, and if $N_J(R) \subset J(R)$, then R is weakly-abelian, which can be obtained as a corollary of the next theorem:

Theorem 3 The following statements are equivalent for a ring R:

(1) R is weakly-abelian;

(2) $aN_J(R)e \subset J(R)$ for each $e \in E(R)$, $a \in A_J(R)$;

(3) $eN_J(R)a \subset J(R)$ for each $e \in E(R)$, $a \in A_J(R)$;

(4) $rN_J(R)e \subset J(R)$ for each $e \in E(R)$, $r \in R$;

(5) $eN_J(R)r \subset J(R)$ for each $e \in E(R)$, $r \in R$.

Proof (1)$\Rightarrow$(2) to (5) is clear by Theorem 1. It is suffice to show (2), (4)$\Rightarrow$(1), and we can get (3), (5)$\Rightarrow$(1) similarly.

(2)$\Rightarrow$(1). For each $a \in A_J(R)$, let

$$ea \equiv a(1-e), e \in E(R),$$

$$a(1-e)re \in aN_J(R)e \subset J(R)$$

for each $r \in R$. Then $0 \equiv a(1-e)re \equiv eare$. By Theorem 4, R is weakly-abelian.

(4)$\Rightarrow$(1). For each $a \in A_J(R)$, $e \in E(R)$, we have $aN_J(R)e \subset J(R)$. Clearly, R is weakly-abelian. □

Corollary 1 The following statements are equivalent for a ring R:

(1) R is weakly-abelian;

(2) $aN(R)e \subset J(R)$ for each $e \in E(R)$, $a \in A_J(R)$;

(3) $eN(R)a \subset J(R)$ for each $e \in E(R)$, $a \in A_J(R)$;

(4) $rN(R)e \subset J(R)$ for each $e \in E(R)$, $r \in R$;

(5) $eN(R)r \subset J(R)$ for each $e \in E(R)$, $r \in R$.

Let $l_J(a) = \{b \mid ba \in J(R)\}$, $r_J(a) = \{b \mid ab \in J(R)\}$. Obviously, $l(a) \subset l_J(a)$, and $r(a) \subset r_J(a)$.

Theorem 4 The following statements are equivalent for a ring R:

(1) R is weakly-abelian;

(2) $al_J(a) \subset J(R)$ for each $a \in A_J(R)$;

(3) $r_J(a)a \subset J(R)$ for each $a \in A_J(R)$.

Proof It is clear that (1) ⇒ (2), (3). It suffice to show (2) ⇒ (1), and (3) ⇒ (1) can be got similarly. (2) ⇒ (1). For each $e \in E(R)$, $r \in R$, $(1-e) \in l(er(1-e))$, thus $er(1-e) \in al(a) \subset J(R)$. □

Corollary 2 The following statements are equivalent for a ring R:

(1) R is weakly-abelian;

(2) $al(a) \subset J(R)$ for each $a \in A_J(R)$;

(3) $r(a)a \subset J(R)$ for each $a \in A_J(R)$.

Theorem 5 The following statements are equivalent for a ring R:

(1) R is weakly-abelian;

(2) $l_J(a) \subset r_J(a)$ for each $a \in A_J(R)$;

(3) $r_J(a) \subset l_J(a)$ for each $a \in A_J(R)$.

Proof It is suffice to show (2) ⇒ (1). For each $e \in E(R)$, $r \in R$, since $er(1-e) \in A_J(R)$, and $1-e \in l_J(er(1-e)) \subset r_J(a)$, thus

$$er(1-e) \in J(R).$$

Therefore R is weakly-ablian. □

Corollary 3 The following statements are equivalent for a ring R:

(1) R is weakly-abelian;

(2) $l(a) \subset r(a)$ for each $a \in A_J(R)$;

(3) $r(a) \subset l(a)$ for each $a \in A_J(R)$;

(4) $l(a) = r(a)$ for each $a \in A_J(R)$;

(5) $l_J(a) = r_J(a)$ for each $a \in A_J(R)$.

Theorem 6 The following statements are equivalent for a ring R:

(1) R is weakly-abelian;

(2) $abc \in J(R)$ implies $bac \in J(R)$ for each $a, b, c \in A_J(R)$;

(3) $ab \in J(R)$ implies $ba \in J(R)$ for each $a, b \in A_J(R)$;

(4) $ab \in J(R)$ implies $aRb \subset J(R)$ for each $a, b \in A_J(R)$.

Proof (1) ⇒ (2), (2) ⇒ (3). Clearly.

(3) ⇒ (4). Let $ab \in J(R)$, then $bar \in J(R)$ for each $r \in R$, thus $arb \in J(R)$.

(4) ⇒ (1). For each $e \in E(R)$, $r \in R$, let $a = er(1-e)$, then $a^2 = 0 \in J(R)$, and $aRa \subset J(R)$, then $a = er(1-e) \in J(R)$. Therefore, R is weakly-abelian. □

Next as the applications of weakly-ablian properties, we will discuss the relations among weakly-abelian rings and other rings. According to Cohn[4], R is called reversible if $ab = 0$ implies $ba = 0$ for each $a, b \in R$. According to Lembek[5], R is called symmetric if $abc = 0$ implies $bac = 0$ for each $a, b, c \in R$. Anderson-Camillo[6] used the term ZC_2 and ZC_3 to denote reversible rings and symmetric rings respectively. It is well known that R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$ implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ over the set $\{1, 2, \cdots, n\}$ and $r_1, r_2, \cdots, r_n \in R$ (Krempa[7]). R is semi-commutative if $ab = 0$ implies $aRb = 0$ for each $a, b \in R$. And R is called reduced ring if $a^2 = 0$ implies $a = 0$ for each $a \in R$. Reduced rings are symmetric, symmetric rings are clearly reversible, reversible rings are semi-commutative, and semi-commutative rings are abelian (Huh[8]).

Theorem 7 Let R be a ring with $N_J(R) \subset A_J(R)$. Then the following statements are equivalent:

(1) R is weakly-abelian;

(2) $R/J(R)$ is reduced;

(3) $R/J(R)$ is symmetric;

(4) $R/J(R)$ is reversible;

(5) $R/J(R)$ is semi-commutative.

Proof (1) ⇒ (2), (2) ⇒ (3), (3) ⇒ (4), (4) ⇒ (5), clearly. It is suffice to show (5) ⇒ (1).

For each $e \in E(R)$, $e(1-e) = 0 \in J(R)$, thus $eR(1-e) \subset J(R)$. Therefore, R is weakly-abelian.

□

Recall that a ring R is called semiabelian, if $eR(1-e) = 0$ or $(1-e)Re = 0$ for each $e \in E(R)$. Then we can get a characterization of abelian rings:

Theorem 8 The following statements are equivalent for a ring R :

(1) R is abelian;

(2) R is semiabelian, and $aRe = 0$ implies $eRa = 0$ for each $e \in E(R)$, $a \in A_J(R)$;

(3) $eA_J(R)(1-e) = 0$ for each $e \in E(R)$.

Proof (1) $\Rightarrow$ (2). Clearly.

(2) $\Rightarrow$ (3). If there exists $e \in E(R)$, $a \in A_J(R)$, $ea(1-e) \neq 0$, then $ea(1-e)Re = 0$ and therefore $eeea(1-e) = ea(1-e) \in eRea(1-e) = 0$ by hypothesis, a contradiction. Hence

$$eA_J(R)(1-e) = 0.$$

(3) $\Rightarrow$ (1). For each $e \in E(R)$, $eR(1-e) = e(eR(1-e))(1-e) \subset eA_J(R)(1-e) = 0$. Hence R is abelian. □

Theorem 9 Let R be a ring with $N_J(R) \subset A_J(R)$. Then the following statements are equivalent:

(1) R is weakly-abelian;

(2) $Rb + R(ba-1) = R$ for each $a \in N_J(R)$, $b \in R$;

(3) For each maximal left ideal M, $a \in N_J(R)$ implies $Ma \subset M$.

Proof (1) $\Rightarrow$ (2). Let $a \in N_J(R)$, $b \in R$. Suppose $Rb + R(ba-1) \neq R$, then there exists a maximal left ideal M such that $Rb + R(ba-1) \subset M$. By Theorem 5, $a \in N_J(R) \subset J(R) \subset M$, then $ba \in M$. Since $ba - 1 \in Rb + R(ba-1) \subset M$, then $1 \in M$, a contradiction. Therefore, $Rb + R(ba-1) = R$.

(2) $\Rightarrow$ (3). If there exist a maximal left ideal M and $a \in N_J(R)$ such that $Ma \not\subset M$, then $M + Ma = R$, and there exist $m, n \in M$ such that $m + na = 1$. By (2), $R = Rm + Rn \subset M$, which is a contradiction. Therefore the conclusion follows.

(3) $\Rightarrow$ (1). If R is not weakly-abelian, then $eR(1-e) \in J(R)$, so there exists a maximal left ideal M and $r \in R$ such that $Mer(1-e) \subset M$. Note that $er(1-e) \in N_J(R)$, by (3), $Mer(1-e) \subset M$, which is a contradiction. Hence, R is weakly-abelian. □

References:

[1] Yu H P. *On quasi-duo rings* [J]. Glasgow Math J, 1995, 37: 21-31.

[2] Chen W X. *On semiabelian π-regular rings* [J]. Intern J Math Sci, 2007, 23: 1-10.

[3] Wei J C. *Some applications of idempotents in ring theory* [D]. Yangzhou: Yangzhou University, 2010.

[4] Cohn P M. *Reversible rings* [J]. Bull London Math Soc, 1999, 31: 641-648.

[5] Lambek J. *On the representation of modules by sheaves of factor moudules* [J]. Canad Math Bull, 1971, 14(3): 359-368.

[6] Anderson D D, Camillo V. *Semigroups and rings whose zero products commute* [J]. Comm Algebra, 1999, 27(6): 2847-2852.

[7] Krempa J, Niewieczerzal D. *Rings in which annihilators are ideals and their application to semigroup rings* [J]. Bull Acad Polon Sci Ser Sci Math Astronom Phys, 1977, 25: 851-856.

[8] Huh C, Lee Y, Smoktunowicz A. *Armendariz rings and semicommutative rings* [J]. Comm Algebra, 2002, 30(2): 751-761.

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(上接第 284 页)

参考文献:

[1] Pei H S. *Equivalences, α-semigroups and α-congruences* [J]. Semigroup Forum, 1994, 49: 49-58.

[2] Pei H S. *Regularity and Green's relations for semigroups of transformations that presereve an equivalence* [J]. Communications in Algebra, 2005, 33: 109-118.

[3] Pei H S. *On the rank of the semigroup $T_E(X)$* [J]. Semigroup Forum, 2005, 70: 107-117.

[4] Pei H S, Sun L, Zhai H C. *Green's relations for the variants of transformation semigroups preserving an equivalence relation* [J]. Communications in Algebra, 2007, 35: 1971-1986.

[5] Pei H S, Zhou H J. *Abundant semigroups of transformations preserving an equivalence relation* [J]. Algebra Colloquium, 2011, 18: 77-82.

[6] Howie J M. *Fundamentals of semigroup theory* [M]. London: Oxford University Press, 1995.

[7] 裴惠生, 吴永福. 保持两个等价关系的夹心半群的格林关系和正则性[J]. 信阳师范学院学报: 自然科学版, 2008, 21(2): 161-164.

[8] 裴惠生, 卢凤梅. 一个变换半群的同余[J]. 信阳师范学院学报: 自然科学版, 2011, 24(3): 281-286.

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