信阳师范学院学报:自然科学版
第 22卷 第 2期 2009年 4月
Journal of Xinyang Normal University
Natural Science Edition Vol 22 No. 2 Apr. 2009

DOI:10.3969/j. issn. 1003-0972. 2009. 02. 001

$|A(G)| = 2^4 p^2 q$的有限交换群 G的构造

张全超[1*],胡 明[2],胡余旺[3]

(1. 武汉大学 东湖分校 高等数学教研室,湖北 武汉 430212;
2. 景德镇高等专科学校 数学与计算机系,江西 景德镇 333000;
3. 信阳师范学院 数学与信息科学学院,河南 信阳 464000)

摘 要:利用有限交换群 G的自同构群 $A(G)$的阶来刻划群 G的结构,证明了当 $|A(G)| = 2^4 p^2 q(p, q$是不同的奇素数)时,G至多有 150型.

关键词:自同构群;交换群;群构造;欧拉函数
中图分类号:O152.1 **文献标志码**:A **文章编号**:1003-0972(2009)02-0161-04

The Structures of Finite Abelian Groups with $|A(G)| = 2^4 p^2 q$

ZHANG Quan-chao[1*], HU Ming[2], HU Yu-wang[3]

(1. Teaching and Research Section of Higher Math., Donghu College, Wuhan University, Wuhan 430212, China;
2. Department of Mathematics and Computer, Jingdezhen Comprehensive College, Jingdezhen 333000, China;
3. College of Math. and Information Science, Xinyang Normal University, Xinyang 464000, China)

Abstract: The structures of Abelian group G was discussed by order of automorphism group $A(G)$ and all types of finite Abelian group G was obtained when the order of $A(G)$ equals to $2^4 p^2 q(p, q$ are different odd primes). The following theorem is proved: let G be finite Abelian group, if $|A(G)| = 2^4 p^2 q(p, q$ are different odd primes), then G has at most 150 types

Key words: automorphism group; Abelian group; structure of group; Euler's function

0 引言及引理

群的构造与分类是群论的重要课题.自从有限单群分类问题解决之后,群论界越来越多的学者对有限群的自同构群的研究产生了极大的兴趣.众所周知,自同构群 $A(G)$由群 G所决定.然而,由 $A(G)$的阶确定 G的结构仍相当复杂.文献[1-4]表明,目前在这方面的研究相当活跃.本文继续上述文献的工作,运用群论的相关结构,结合数论知识,采取枚举讨论的办法,解决了 $|A(G)| = 2^4 p^2 q(p, q$是不同的奇素数)的有限 Abelian群的构造问题.

引理 1[5] 有限交换群 G可唯一分解为其循环西洛子群的直积.

引理 2[5] 若群 $G \cong C_n$,则 $|A(G)| = \varphi(n)$;若 G为 p^n阶交换群,则 $p^{n-1}(p-1) \mid |A(G)|$.

引理 3[5] 设 $G = H \times K$,则当 $(|H|, |K|) = 1$时,$A(G) = A(H) \times A(K)$.

引理 4[6] 设 G是 p^n阶交换群,G的型为

$$[\underbrace{m_1, \cdots, m_1}_{s_1}, \underbrace{m_2, \cdots, m_2}_{s_2}, \cdots, \underbrace{m_t, \cdots, m_t}_{s_t}],$$

其中 $m_1 > m_2 > \cdots > m_t$, $s_i > 0$,则

$$|A(G)| = p^u \prod_{i=1}^{t} \prod_{k=1}^{s_i} (p^k - 1),$$

其中

$$u = \sum_{i,j=1}^{t} s_i s_j m_{ij} - \sum_{i=1}^{t} \frac{s_i(s_i+1)}{2}, m_{ij} = m_{\max\{i,j\}}.$$

1 主要结果及证明

下面定理的证明过程中,我们总假定 $p_1, \cdots, p_k$是互异的奇素数.

定理 1 设群 G是有限交换群,$|A(G)| = 2^4 p^2$

收稿日期:2008-03-26;修订日期:2008-10-30; *.通讯联系人,E-mail: zhangjichu2006@163.com
基金项目:河南省自然科学基金项目(0511010200)
作者简介:张全超(1983-),男,河南洛阳人,助教,硕士,主要从事有限群的研究.

q(p, q为互异的奇素数),则 G最多有 150型.

证明 设 $|G| = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$,于是 $G \cong S_2 \times S_{p_1} \times S_{p_2} \times \cdots \times S_{p_k}$, $A(G) \cong A(S_2) \times A(S_{p_1}) \times A(S_{p_2}) \times \cdots \times A(S_{p_k})$且 $|A(G)| = |A(S_2)| \cdot |A(S_{p_1})| \cdot |A(S_{p_2})| \cdots |A(S_{p_k})|$. 由引理 2,对于每一个奇素数 p_i, $2 \mid |A(S_{p_i})| \mid |A(G)| \Rightarrow 2^k \mid |A(G)| = 2^4 p^2 q$,所以 $k = 0, 1, 2, 3, 4$.

(1) $k=0$时,经计算,群 G不存在.

(2) $k=1$时, $|G| = 2^{\alpha_0} p_1^{\alpha_1}$,因为 $2 \mid |A(S_{p_1})|$,所以 $\alpha_0 = 0, 1, 2, 3, 4$.

当 $\alpha_0 = 0, 1$时, $A(S_2) = 1$,由 $p_1^{\alpha_1 - 1}(p_1 - 1) \mid 2^4 p^2 q$,知 $\alpha_1 = 1, 2, 3$. 当 $\alpha_1 = 1$时, $G \cong C_{p_1}, C_2 \times C_{p_1}$, $|A(G)| = p_1 - 1 = 2^4 p^2 q \Rightarrow p_1 = 2^4 p^2 q + 1$,当 $2^4 p^2 q + 1$为素数时,有群 $G_1 \cong C_{2^4p^2q+1}$, $G_2 \cong C_2 \times C_{2^4p^2q+1}$;当 $\alpha_1 = 2$时, $S_{p_1} \cong C_{p_1^2}, C_{p_1} \times C_{p_1}$. 若 $S_{p_1} \cong C_{p_1^2}$,则 $G \cong C_{p_1^2}, C_2 \times C_{p_1^2}$, $|A(G)| = p_1(p_1 - 1) = 2^4 p^2 q$, $\Rightarrow p_1 = q = 2^4 p^2 + 1$,若 $2^4 p^2 + 1$为素数,则有群 $G_3 \cong C_{(2^4p^2+1)^2}$, $G_4 \cong C_2 \times C_{(2^4p^2+1)^2}$. 若 $S_{p_1} \cong C_{p_1} \times C_{p_1}$,经计算群 G不存在;当 $\alpha_1 = 3$时, $S_{p_1} \cong C_{p_1^3}, C_{p_1^2} \times C_{p_1}, C_{p_1} \times C_{p_1} \times C_{p_1}$. 若 $S_{p_1} \cong C_{p^3}$,则 $G \cong C_{p_1^3}, C_2 \times C_{p_1^3}$, $\Rightarrow |A(G)| = p_1^2(p_1 - 1) = 2^4 p^2 q$, $\Rightarrow p_1 = p = 2^4 q + 1$,若 $2^4 q + 1$为素数,则有群 $G_5 \cong C_{(2^4q+1)^3}$, $G_6 \cong C_2 \times C_{(2^4q+1)^3}$. 其他情况经计算群 G不存在.

当 $\alpha_0 = 2$时, $S_2 \cong C_{2^2}, C_2 \times C_2$. 当 $\alpha_1 = 1$时, $S_{p_1} \cong C_{p_1}$. 若 $S_2 \cong C_{2^2}$,则 $G \cong C_{2^2} \times C_{p_1}$, $\Rightarrow |A(G)| = 2(p_1 - 1) = 2^4 p^2 q \Rightarrow p_1 = 2^3 p^2 q + 1$. 若 $2^3 p^2 q + 1$为素数,则有群 $G_7 \cong C_{2^2} \times C_{2^3p^2q+1}$. 若 $S_2 \cong C_2 \times C_2$,则 $G \cong C_2 \times C_2 \times C_{p_1}$, $\Rightarrow |A(G)| = 2 \cdot 3 \cdot (p_1 - 1) = 2^4 p^2 q$ ①若 $q = 3$, $\Rightarrow p_1 = 2^3 p^2 + 1$,则当 $2^3 p^2 + 1$为素数时,有群 $G_8 \cong C_2 \times C_2 \times C_{2^3p^2+1}$. ②若 $p = 3$, $\Rightarrow p_1 = 24q + 1$,则当 $24q + 1$为素数时,有群 $G_9 \cong C_2 \times C_2 \times C_{24q+1}$;当 $\alpha_1 = 2$时, $S_{p_1} \cong C_{p_1^2}, C_{p_1} \times C_{p_1}$. 若 $S_2 \cong C_{2^2}$, $\Rightarrow$(*) $G \cong C_{2^2} \times C_{p_1^2}$或(**) $G \cong C_{2^2} \times C_{p_1} \times C_{p_1}$. 由(*) $\Rightarrow |A(G)| = 2p_1(p_1 - 1) = 2^4 p^2 q$, $\Rightarrow p_1 = q = 2^3 p^2 + 1$,当 $2^3 p^2 + 1$为素数时,有群 $G_{10} \cong C_{2^2} \times C_{(2^3p^2+1)^2}$. 由(**) $\Rightarrow |A(G)| = 2p_1(p_1 - 1)^2(p_1 + 1) = 2^4 p^2 q$,不可能成立;若 $S_2 \cong C_2 \times C_2$,经计算群 G不存在;当 $\alpha_1 = 3$时, $S_{p_1} \cong C_{p_1^3}, C_{p_1^2} \times C_{p_1}, C_{p_1} \times C_{p_1} \times C_{p_1}$. 若 $G \cong C_{2^2} \times C_{p_1^3}$,则 $|A(G)| = 2(p_1 - 1)p_1^2 = 2^4 p^2 q$, $\Rightarrow (p_1 - 1)p_1^2 = 2^3 p^2 q \Rightarrow p_1 = p = 2^3 q + 1$,当 $2^3 q + 1$为素数时,有群 $G_{11} \cong C_{2^2} \times C_{(2^3q+1)^3}$. 其他情况经计算群不存在.

当 $\alpha_0 = 3$时, $S_2 \cong C_{2^3}, C_{2^2} \times C_2, C_2 \times C_2 \times C_2$. 当 $\alpha_1 = 1$时, $S_{p_1} \cong C_{p_1}$. 若 $G \cong C_{2^3} \times C_{p_1}$,则 $|A(G)| = 2^2(p_1 - 1) = 2^4 p^2 q \Rightarrow p_1 = 2^2 p^2 q + 1$,若 $2^2 p^2 q + 1$为素数,则有群 $G_{12} \cong C_{2^3} \times C_{2^2p^2q+1}$. 若 $G \cong C_{2^2} \times C_2 \times C_{p_1}$,则 $|A(G)| = 2^3(p_1 - 1) = 2^4 p^2 q \Rightarrow p_1 = 2p^2 q + 1$,若 $2p^2 q + 1$为素数,则有群 $G_{13} \cong C_{2^2} \times C_2 \times C_{2p^2q+1}$. 若 $G \cong C_2 \times C_2 \times C_2 \times C_{p_1}$,则 $|A(G)| = 2^3 \cdot 3 \cdot 7(p_1 - 1) = 2^4 p^2 q$ 经计算有群 $G_{14} \cong C_2 \times C_2 \times C_2 \times C_7$;当 $\alpha_1 = 2$时, $S_{p_1} \cong C_{p_1^2}, C_{p_1} \times C_{p_1}$. 若 $G \cong C_{2^3} \times C_{p_1^2}$,则 $|A(G)| = 2^2 p_1(p_1 - 1) = 2^4 p^2 q$ 经计算有群 $G_{15} \cong C_{2^3} \times C_{(2^2p^2+1)^2}$. 若 $G \cong C_{2^2} \times C_2 \times C_{p_1^2}$,则 $|A(G)| = 2^3 p_1(p_1 - 1) = 2^4 p^2 q$ 经计算,当 $2p^2 + 1$为素数时,有群 $G_{16} \cong C_{2^2} \times C_2 \times C_{(2p^2+1)^2}$. 若 $G \cong C_2 \times C_2 \times C_2 \times C_{p_1^2}$,则 $|A(G)| = 2^3 \cdot 3 \cdot 7(p_1 - 1)p_1 = 2^4 p^2 q$ 经计算有群 $G_{17} \cong C_2 \times C_2 \times C_2 \times C_{3^2}$. 其他情况经计算群 G不存在;当 $\alpha_1 = 3$时, $S_{p_1} \cong C_{p_1^3}, C_{p_1^2} \times C_{p_1}, C_{p_1} \times C_{p_1} \times C_{p_1}$. 若 $G \cong C_{2^3} \times C_{p_1^3}$,则 $|A(G)| = 2^2 p_1^2(p_1 - 1) = 2^4 p^2 q \Rightarrow p_1^2(p_1 - 1) = 2^2 p^2 q \Rightarrow p_1 = q = 2^2 q + 1$,所以当 $2^2 q + 1$为素数时,有群 $G_{18} \cong C_{2^3} \times C_{(2^2q+1)^3}$. 若 $G \cong C_{2^2} \times C_2 \times C_{p_1^3}$,则 $|A(G)| = 2^3 p_1^2(p_1 - 1) = 2^4 p^2 q \Rightarrow p_1^2(p_1 - 1) = 2p^2 q \Rightarrow p_1 = p = 2q + 1$. 若 $2q + 1$为素数,则有群 $G_{19} \cong C_{2^2} \times C_2 \times C_{(2q+1)^3}$. 其他情况经计算群 G不存在.

当 $\alpha_0 = 4$时,只需讨论 $S_2 \cong C_{2^4}$的情况. 当 $\alpha_1 = 1$时, $G \cong C_{2^4} \times C_{p_1}$, $|A(G)| = 2^3(p_1 - 1) = 2^4 p^2 q \Rightarrow p_1 = 2p^2 q + 1$,若 $2p^2 q + 1$为素数,则有群 $G_{20} \cong C_{2^4} \times C_{2p^2q+1}$;当 $\alpha_1 = 2$时,只须讨论 $S_{p_1} \cong C_{p_1^2}$的情况. 此时 $G \cong C_{2^4} \times C_{p_1^2}$, $\Rightarrow |A(G)| = 2^3 p_1(p_1 - 1) = 2^4 p^2 q \Rightarrow p_1 = q = 2p^2 + 1$,所以当 $2p^2 + 1$为素数时,有群 $G_{21} \cong C_{2^4} \times C_{(2p^2+1)^2}$;当 $\alpha_1 = 3$时,只需讨论 $S_{p_1} \cong C_{p_1^3}$的情况. 此时 $G \cong C_{2^4} \times C_{p_1^3}$, $\Rightarrow |A(G)| = 2^3 p_1^2(p_1 - 1) = 2^4 p^2 q \Rightarrow p_1 = p = 2q + 1$,所以当 $2q + 1$为素数时,有群 $G_{22} \cong C_{2^4} \times C_{(2q+1)^3}$.

(3)当 $k = 2$时, $\alpha_0 = 0, 1, 2, 3$. 由 $p_1^{\alpha_1}(p_1 - 1) p_2^{\alpha_2}(p_2 - 1) \mid 2^4 p^2 q$,知 $\alpha_1 = 1, 2, 3$; $\alpha_2 = 1, 2, 3$.

当 $\alpha_0 = 0, 1$时,只须考虑:当 $(\alpha_1, \alpha_2) = (1, 1)$时, $G \cong C_{p_1} \times C_{p_2}, C_2 \times C_{p_1} \times C_{p_2}$, $\Rightarrow |A(G)| = (p_1 - 1)(p_2 - 1)$. 由于 $(p_1 - 1)$和 $(p_2 - 1)$是不同的偶数,而 $2^4 p^2 q = 2 \cdot 2^3 p^2 q = 2q \cdot 2^3 p^2 = 2p \cdot 2^3 pq = 2p^2 \cdot 2^3 q = 2pq \cdot 2^3 p = 2p^2 q \cdot 2^3 = 2^2 \cdot 2^2 p^2 q = 2^2 q \cdot 2^2 p^2 = 2^2 p \cdot 2^2 pq$ 经计算,并考虑到 p_1和 p_2的对称性,当 $2^3 p^2 q + 1$, $2q + 1$, $2^3 p^2 + 1$, $2p + 1$, $2^3 pq + 1$,

$2p^2+1, 2^3q+1, 2pq+1, 2^3p+1, 2^2p^2q+1, 2^2p^2+1, 2^2q+1, 2^2p+1, 2^2pq+1$为素数时,有群 $G_{23}\sim G_{38}$,计 16型.

当$(\alpha_1,\alpha_2)=(2,1)$时,若 $G\cong C_{p_1^2}\times C_{p_2}, C_2\times C_{p_1^2}\times C_{p_2}$,则$|A(G)|=p_1(p_1-1)(p_2-1)=2^4p^2q$.①当 $p_1=q$时,$(q-1)(p_2-1)=2^4p^2$,而 $2^4p^2=2\cdot2^3p^2=2p\cdot2^3p=2^2\cdot2^2p^2=2p^2\cdot2^3$.经计算,并考虑到 p_1到 p_2的不对称性,当 $2^3p^2+1, 2p+1, 2^3p+1, 2^2p^2+1$为素数时,有群 $G_{39}\sim G_{50}$,计 12型.②当 $p_1=p$时,$(p-1)(p_2-1)=2^4pq$,而 $2^4pq=2\cdot2^3pq=2p\cdot2^3q=2q\cdot2^3p=2pq\cdot2^3=2^2\cdot2^2pq=2^2p\cdot2^2q$.经计算,并考虑到 p_1和 p_2的不对称性,当 $2^3pq+1, 2p+1, 2^3q+1, 2q+1, 2^3p+1, 2^2pq+1$为素数时,有群 $G_{51}\sim G_{60}$,计 10型.若 $G\cong C_{p_1}\times C_{p_1}\times C_{p_2}, C_2\times C_{p_1}\times C_{p_1}\times C_{p_2}$,则群 G不存在.

当$(\alpha_1,\alpha_2)=(3,1)$时,此时只须考虑 $S_{p_1}\cong C_{p_1^3}, C_{p_1^2}\times C_{p_1}$的情况.若 $G\cong C_{p_1^3}\times C_{p_2}, C_2\times C_{p_1^3}\times C_{p_2}$,则$|A(G)|=p_1^2(p_1-1)(p_2-1)=2^4p^2q$,$\Rightarrow p_1=p$,$(p-1)(p_2-1)=2^4q$,而 $2^4q=2\cdot2^3q=2\cdot2^2q$.经计算,并考虑到 p_1和 p_2的不对称性,当 $2^3q+1, 2^2q+1$为素数时,有群 $G_{61}\sim G_{68}$,计 8型.当 G为其他情况时,经计算不存在.

当$(\alpha_1,\alpha_2)=(2,2)$时,若 $G\cong C_{p_1^2}\times C_{p_2^2}$或 $G\cong C_2\times C_{p_1^2}\times C_{p_2^2}$,则$|A(G)|=p_1(p_1-1)p_2(p_2-1)=2^4p^2q\Rightarrow p_1=p, p_2=q$,$(p-1)(q-1)=2^4p$,而 $2^4p=2\cdot2^3p=2^2\cdot2^2p=2^3\cdot2p$.经计算,并考虑到 p_1和 p_2的对称性,当 $2^3p+1, 2^2p+1$为素数时,有群 $G_{69}\sim G_{70}$,计 2型.$G_{69}\cong C_{3^2}\times C_{(2^3p+1)^2}$,$G_{70}\cong C_{5^2}\times C_{(2^2p+1)^2}$.其他情况经计算群 G不存在.

当$(\alpha_1,\alpha_2)=(3,2)$时,经计算群 G不存在.

当 $\alpha_0=2$时,$S_2\cong C_{2^2}, C_2\times C_2$.当$(\alpha_1,\alpha_2)=(1,1)$时,若 $G\cong C_{2^2}\times C_{p_1}\times C_{p_2}$,$\Rightarrow|A(G)|=2(p_1-1)(p_2-1)=2^4p^2q$,$\Rightarrow(p_1-1)(p_2-1)=2^3p^2q$.而 $2^3p^2q=2\cdot2^2p^2q=2q\cdot2^2p^2=2p\cdot2^2pq=2p^2\cdot2^2q=2pq\cdot2^2p=2p^2q\cdot2^2$.当 $2^2p^2q+1, 2q+1, 2^2p^2+1, 2p+1, 2^2pq+1, 2p^2+1, 2^2q+1, 2pq+1, 2^2p+1, 2p^2q+1$为素数时,有群 $G_{71}\sim G_{76}$,计 6型.若 $G\cong C_2\times C_2\times C_{p_1}\times C_{p_2}$,$\Rightarrow|A(G)|=2\cdot3(p_1-1)(p_2-1)=2^4p^2q$.①若 $q=3$,则$(p_1-1)(p_2-1)=2^3p^2$,而 $2^3p^2=2\cdot2^2p^2=2p\cdot2^2p=2p^2\cdot2^2$.当 $2^2p^2+1, 2p+1, 2^2p+1, 2p^2+1$为素数时,有群 $G_{77}\sim G_{79}$,计 3型.$G_{77}\cong C_2\times C_2\times C_3\times C_{2^2p^2+1}$,$G_{78}\cong C_2\times C_{2p+1}\times C_{2^2p+1}$,$G_{79}\cong C_2\times C_2\times C_{2p^2+1}\times C_5$.②若 $p=3$,则$(p_1-1)(p_2-1)=2^3\cdot3q$,而 $2^3\cdot3q=2\cdot12q=6\cdot4q=2q\cdot12=6q\cdot2^2$.当 $12q+1, 4q+1, 2q+1, 6q+1$为素数时,有群 $G_{80}\sim G_{83}$,计 4型.

当$(\alpha_1,\alpha_2)=(1,2)$时,只须考虑 $S_{p_2}\cong C_{p_2^2}$的情况.若 $G\cong C_{2^2}\times C_{p_1}\times C_{p_2^2}$,$\Rightarrow|A(G)|=2(p_1-1)p_2(p_2-1)=2^4p^2q$.①当 $p_2=q$时,$\Rightarrow(p_1-1)(q-1)=2^3p^2$.而 $2^3p^2=2\cdot2^2p^2=2p\cdot2^2p=2p^2\cdot2^2$.当 $2^2p^2+1, 2p+1, 2^2p+1, 2p^2+1$为素数时,有群 $G_{84}\sim G_{89}$,计 6型.②当 $p_2=p$时,$\Rightarrow(p_1-1)(p-1)=2^3pq$.而 $2^3pq=2\cdot2^2pq=2p\cdot2^2q=2q\cdot2^2p=2pq\cdot2^2$.当 $2p+1, 2^2q+1, 2pq+1, 2^2pq+1, 2^2p+1, 2q+1$为素数时,有群 $G_{90}\sim G_{93}$,计 4型.$G_{90}\cong C_{2^2}\times C_{2p+1}\times C_{(2^2q+1)^2}$,$G_{91}\cong C_{2^2}\times C_{2pq+1}\times C_{5^2}$,$G_{92}\cong C_{2^2}\times C_{2^2pq+1}\times C_{3^2}$,$G_{93}\cong C_{2^2}\times C_5\times C_{(2pq+1)^2}$.若 $G\cong C_2\times C_2\times C_{p_1}\times C_{p_2^2}$,$|A(G)|=2\cdot3(p_1-1)p_2(p_2-1)=2^4p^2q$.①当 $q=3$时,$\Rightarrow(p_1-1)(p_2-1)p_2=2^3p^2$,$\Rightarrow p_1=11, p_2=5, p=5$,有群 $G_{94}\cong C_2\times C_2\times C_{11}\times C_{5^2}$.②当 $p=3$时,$\Rightarrow(p_1-1)p_2(p_2-1)=2^3\cdot3q$.若 $p_2=3$, $p_1-1=2^2q$, $p_1=4q+1$.若 $p_2=q$, $(p_1-1)(q-1)=2^3\cdot3$.$\Rightarrow(p_1,p_2)=(3,13)$,或$(p_1,p_2)=(7,5)$.当 $4q+1$为素数时,有群 $G_{95}\sim G_{100}$,计 6型.

当$(\alpha_1,\alpha_2)=(1,3)$时,只须考虑 $S_{p_2}\cong C_{p_2^3}$的情况.若 $G\cong C_{2^2}\times C_{p_1}\times C_{p_2^3}$,则$|A(G)|=2(p_1-1)p_2^2(p_2-1)=2^4p^2q\Rightarrow(p_1,p_2)=(2,4q+1)$或$(p_1,p_2)=(2q+1,5)$.因此当 $2q+1, 4q+1$为素数时,有群 $G_{101}\sim G_{104}$,计 4型.若 $G\cong C_2\times C_2\times C_{p_1}\times C_{p_2^3}$,则$|A(G)|=2\cdot3(p_1-1)p_2^2(p_2-1)=2^4p^2q\Rightarrow(p_1,p_2)=(3,5)$.因此有群 $G_{105}\sim G_{106}$,计 2型.

当$(\alpha_1,\alpha_2)=(2,1)$时,由对称性,还是群 $G_{84}\sim G_{100}$,另加群 $G_{107}\cong C_2\times C_2\times C_5\times C_{11^2}$(与群 G_{94}相对称).

当$(\alpha_1,\alpha_2)=(2,2)$时,只须考虑:若 $G\cong C_{2^2}\times C_{p_1^2}\times C_{p_2^2}$,则$|A(G)|=2p_1(p_1-1)p_2(p_2-1)=2^4p^2q$.①$p_1=p$时,$\Rightarrow(p_1,p_2)=(3,2^2p+1)$;②$p_1=q$时,$\Rightarrow(p_1,p_2)=(2p+1,5)$;③$p_2=p$时,$\Rightarrow(p_1,p_2)=(2p+1,5)$;④$p_2=q$时,$\Rightarrow(p_1,p_2)=(3,2^2p+1)$.因此,当 $2p+1, 2^2p+1$为素数时,有群 $G_{108}\sim G_{109}$,计 2型.$G_{108}\cong C_{2^2}\times C_{(2p+1)^2}\times C_{5^2}$,$G_{109}\cong C_{2^2}\times C_{3^2}\times C_{(2^2p+1)^2}$.若 $G\cong C_2\times C_2\times C_{p_1^2}\times C_{p_2^2}$,则$|A(G)|=2\cdot3p_1(p_1-1)p_2(p_2-1)=2^4p^2q$.经计算$(p_1,p_2)=(3,5)$,因此有群 $G_{110}\cong C_2\times C_2\times C_{3^2}\times$

C_{5^2}.

当 $(\alpha_1,\alpha_2)=(2,3)$时,只须考虑:若 $G\cong C_{2^2}\times C_{p_1^2}\times C_{p_2^3}$,则 $|A(G)|=2p_1(p_1-1)(p_2-1)p_2^2=2^4p^2q$, $\Rightarrow(p_1,p_2)=(3,5)$,或 $(p_1,p_2)=(5,3)$. 因此有群 $G_{111}\sim G_{112}$,计 2型. $G_{111}\cong C_{2^2}\times C_{3^2}\times C_{5^3}$, $G_{112}\cong C_{2^2}\times C_{5^2}\times C_{3^3}$. 若 $G\cong C_2\times C_2\times C_{p_1^2}\times C_{p_2^3}$,经计算群 G不存在.

当 $\alpha_0=3$时,只须考虑:当 $(\alpha_1,\alpha_2)=(1,1)$时,只须考虑 $S_2\cong C_2^3$的情况. 此时, $|A(G)|=2^2(p_1-1)(p_2-1)=2^4p^2q$, $\Rightarrow(p_1-1)(p_2-1)=2^2p^2q$而 $2^2p^2q=2\cdot2p^2q=2p\cdot2pq=2q\cdot2p^2$. 当 $2p+1, 2q+1, 2p^2q+1, 2pq+1, 2p^2+1$为素数时,有群 $G_{113}\sim G_{115}$,计 3型.

当 $(\alpha_1,\alpha_2)=(1,2)$时,只须考虑 $G\cong C_{2^3}\times C_{p_1}\times C_{p_2^2}$的情况. 此时, $|A(G)|=2^2(p_1-1)p_2(p_2-1)=2^4p^2q$ ①当 $p_2=q$时, $\Rightarrow(p_1,p_2)=(3,2p^2+1)$或 $(p_1,p_2)=(2p^2+1,3)$. 当 $2p^2+1$为素数时,有群 $G_{116}\sim G_{117}$,计 2型. ②当 $p_2=p$时, $\Rightarrow(p_1,p_2)=(2pq+1,3)$或 $(p_1,p_2)=(2p+1,2q+1)$. 当 $2p+1, 2q+1, 2pq+1$为素数时,有群 $G_{118}\sim G_{119}$,计 2型.

当 $(\alpha_1,\alpha_2)=(1,3)$时,只须考虑 $G\cong C_{2^3}\times C_{p_1}\times C_{p_2^3}$的情况. 此时, $|A(G)|=2^2(p_1-1)p_2^2(p_2-1)=2^4p^2q\Rightarrow p=p_2$, $(p_1-1)(p-1)=2^2q$,而 $2^2q=2\cdot2q$, $\Rightarrow(p_1,p_2)=(3,2q+1)$或 $(p_1,p_2)=(2q+1,3)$. 当 $2q+1$为素数时,有群 $G_{120}\sim G_{121}$,计 2型. $G_{120}\cong C_{2^3}\times C_3\times C_{(2q+1)^3}$, $G_{121}\cong C_{2^3}\times C_{2q+1}\times C_{3^3}$.

当 $(\alpha_1,\alpha_2)=(2,2)$时,只须考虑 $G\cong C_{2^3}\times C_{p_1^2}\times C_{p_2^2}$的情况. 此时, $|A(G)|=2^2p_1(p_1-1)p_2(p_2-1)=2^4p^2q$ ①$p_1=p$时, $\Rightarrow(p_1,p_2)=(3,2p+1)$. ②$p_1=q$时, $\Rightarrow(p_1,p_2)=(2p+1,3)$. 由 p_1,p_2的对称性,知当 $2p+1$为素数时,有群 $G_{122}\cong C_{2^3}\times C_{(2p+1)^2}\times C_{3^2}$.

当 $(\alpha_1,\alpha_2)=(2,3)$时,经计算群 G不存在.

(4)当 $k=3$时, $\alpha_0=0,1,2$.

当 $\alpha_0=0,1$时,当 $(\alpha_1,\alpha_2,\alpha_3)=(1,1,1)$时, $G\cong C_{p_1}\times C_{p_2}\times C_{p_3}$或 $G\cong C_2\times C_{p_1}\times C_{p_2}\times C_{p_3}$. $|A(G)|=(p_1-1)(p_2-1)(p_3-1)=2^4p^2q$,而 $2^4p^2q=2^2p\cdot2p\cdot2q=2^2p\cdot2pq\cdot2=2pq\cdot2p\cdot2^2=2pq\cdot2^2p\cdot2=2p^2\cdot2q\cdot2^2=2p^2\cdot2^2q\cdot2=2^2p^2\cdot2q\cdot2=2^2pq\cdot2p\cdot=2p^2q\cdot2^2\cdot2=2^2q\cdot2p^2\cdot2$ 经计算,当 $2p+1, 4p+1, 2q+1, 4q+1, 2pq+1, 4pq+1, 2p^2+1, 4p^2+1, 2p^2q+1$为素数时,有群 $G_{123}\sim G_{142}$,计 20型.

当 $(\alpha_1,\alpha_2,\alpha_3)=(2,1,1)$时, $G\cong C_{p_1^2}\times C_{p_2}\times C_{p_3}$, $C_2\times C_{p_1^2}\times C_{p_2}\times C_{p_3}$. $|A(G)|=p_1(p_1-1)(p_2-1)(p_3-1)=2^4p^2q$ ①若 $p_1=p$,则 $(p-1)(p_2-1)(p_3-1)=2^4pq$而 $2^4pq=2^2\cdot2p\cdot2q=2p\cdot2q\cdot2^2=2q\cdot2p\cdot2^2=2^2p\cdot2q\cdot2=2q\cdot2^2p\cdot2=2\cdot2^2p\cdot q=2^2q\cdot2p\cdot2=2p\cdot2^2q\cdot2=2\cdot2p\cdot2^2q=2pq\cdot2^2\cdot2=2^2\cdot2pq\cdot2=2\cdot2pq\cdot2^2$. 经计算,当 $2p+1, 4p+1, 2q+1, 4q+1, 2pq+1, 2^2q^2+1$为素数时,有群 $G_{143}\sim G_{156}$,计 14型. ②若 $p_1=q$,则 $(q-1)(p_2-1)(p_3-1)=2^4p^2$. 而 $2^4p^2=2^2p\cdot2p\cdot2=2p^2\cdot2^2\cdot2=2p\cdot2^2p\cdot2=2\cdot2^2p\cdot2p=2^2\cdot2p^2\cdot2=2\cdot2p^2\cdot2^2$. 经计算,当 $2p+1, 4p+1, 2p^2+1$为素数时,有群 $G_{157}\sim G_{168}$,计 12型.

当 $(\alpha_1,\alpha_2,\alpha_3)=(2,2,1)$时, $G\cong C_{p_1^2}\times C_{p_2^2}\times C_{p_3}$, $C_2\times C_{p_1^2}\times C_{p_2^2}\times C_{p_3}$. $|A(G)|=p_1(p_1-1)p_2(p_2-1)(p_3-1)=2^4p^2q$ ①若 $p_1=p$,则 $(p-1)((p_2-1)p_2(p_3-1)=2^4pq\Rightarrow p_2=q$, $(p-1)(q-1)(p_3-1)=2^4p$而 $2^4p=2\cdot2^2\cdot2p=2^2\cdot2p\cdot2=2\cdot2p\cdot2^2$. 经计算,当 $2p+1$为素数时,有群 $G_{169}\sim G_{174}$,计 6型. ②若 $p_1=q$,则 $(q-1)(p_2-1)p_2(p_3-1)=2^4p^2$. $\Rightarrow p_2=p$, $(q-1)(p-1)(p_3-1)=2^4p$而 $2^4p=2^2\cdot2\cdot2p=2^2\cdot2p\cdot2=2p\cdot2\cdot2^2$. 经计算,有群 $G_{175}\sim G_{178}$,计 4型.

当 $(\alpha_1,\alpha_2,\alpha_3)=(2,2,2)$时,经计算群 G不存在.

当 $(\alpha_1,\alpha_2,\alpha_3)=(1,1,3)$时, $G\cong C_{p_1}\times C_{p_2}\times C_{p_3^3}$, $C_2\times C_{p_1}\times C_{p_2}\times C_{p_3^3}$. $|A(G)|=(p_1-1)(p_2-1)p_3^2(p_3-1)=2^4p^2q$, $\Rightarrow p_3=p$, $(p_1-1)(p_2-1)(p-1)=2^4q$而 $2^4q=2^2\cdot2q\cdot2=2\cdot2^2\cdot2q=2\cdot2q\cdot2^2$. 经计算,当 $2q+1$为素数时,有群 $G_{179}\sim G_{184}$,计 6型.

当 $(\alpha_1,\alpha_2,\alpha_3)$为其他组合时,经计算群 G不存在.

当 $\alpha_0=2$时, $S_2\cong C_{2^2}$, $C_2\times C_2$. 当 $(\alpha_1,\alpha_2,\alpha_3)=(1,1,1)$时,若 $G\cong C_{2^2}\times C_{p_1}\times C_{p_2}\times C_{p_3}$,则 $|A(G)|=2(p_1-1)(p_2-1)(p_3-1)=2^4p^2q\Rightarrow(p_1-1)(p_2-1)(p_3-1)=2^3p^2q=2pq\cdot2p\cdot2=2p^2\cdot2q\cdot2$ 经计算,当 $2pq+1, 2p+1, 2q+1, 2p^2+1$为素数时,有群 $G_{185}\sim G_{186}$,计 2型. $G_{185}\cong C_{2^2}\times C_{2pq+1}\times C_{2p+1}\times C_3$, $G_{186}\cong C_{2^2}\times C_{2p^2+1}\times C_{2q+1}\times C_3$. 若 $G\cong C_2\times C_2\times C_{p_1}\times C_{p_2}\times C_{p_3}$, (下转第 171页)

$d = {}^{\perp L}e = e^{\perp R}$,下证 d为对偶元. 由命题 3知 $\forall a \in Q$, $(a \rightarrow_r d) \rightarrow_l d = {}^{\perp L}((a \rightarrow_r d) \& d^{\perp R}) = {}^{\perp L}((a \rightarrow_r d) \& e) = {}^{\perp L}(a \rightarrow_r d) = {}^{\perp L}(({}^{\perp L}d \& d)^{\perp R}) = {}^{\perp L}((e\&a)^{\perp R}) = {}^{\perp L}(a^{\perp R}) = a$; $(a \rightarrow_l d) \rightarrow_r d = ({}^{\perp L}d \& (a \rightarrow_l d))^{\perp R} = ({}^{\perp L}d \& {}^{\perp L}(a \& d^{\perp R}))^{\perp R} = (e \& {}^{\perp L}(a\&e))^{\perp R} = ({}^{\perp L}a)^{\perp R} = a$,所以 $(a \rightarrow_l d) \rightarrow_r d = a = (a \rightarrow_r d) \rightarrow_l d$,即 d为对偶元,Q为对偶 Quantale.

另外,任意 Quantale都可视为其自身上的双重模. 而对偶 Quantale中的前置非 ${}^{\perp L}$与后置非 ${}^{\perp R}$刚好作成拟对偶运算对,而由文献[7]的命题 4知 $a\&m = (a \rightarrow_l {}^{\perp L}m)^{\perp L}m)^{\perp R}$成立,从而对偶 Quantale是其自身上的对偶双重模.

2 总结

本文主要研究了左 Quantale模的余核映射及其性质,对于右 Quantale模,Quantale双重模可有类似的讨论. 最后通过对 Quantale上的对偶双重模的讨论,我们得到了比较好的性质.

参考文献:

[1] Mulvey C J, Pelletier J W. *On the quantisation of point*[J]. Journal of Pure and Applical Algebra(S0022-4049), 2001, 159(2/3): 231-295.

[2] Rosenthal K I *Quantales and their applications*[M]. London: Longman Scientific and Technical, 1990.

[3] Joyal A, Tierney M. *An extension of the galois theory of grothendieck*[J]. Memoirs of the American Mathematical Society(S0065-9266), 1984, 309.

[4] Abramsky S, Vickers S. *Quantales, observational logic and process semantics*[J]. Mathematical Structures in Computer Science(S0960-1295), 1993, 3(2): 161-227.

[5] Kruml D. *Spatial quantales*[J]. Applied Categorial Structures(S0927-2852), 2002, 10(1): 49-62.

[6] Paseka J. *A note on Girard bimodules*[J]. International Journal of Theoretical Physics(S0020-7748), 2000, 39(3): 805-812.

[7] 李永明. 对偶 Quantale及其性质[J]. 陕西师范大学学报:自然科学版, 2001, 29(1): 1-5.

责任编辑:郭红建

(上接第 164页)

则 $|A(G)| = 2 \cdot 3(p_1 - 1)(p_2 - 1)(p_3 - 1) = 2^4p^2q$ ①若 $q = 3$, $(p_1 - 1)(p_2 - 1)(p_3 - 1) = 2^3p^2$,矛盾. ②若 $p = 3$, $(p_1 - 1)(p_2 - 1)(p_3 - 1) = 2^3 \cdot 3q = 2^3 \cdot 3 \cdot q = 2 \cdot 6 \cdot 2q$,经计算,当 $2q + 1$为素数时,有群 $G_{187} \cong C_2 \times C_2 \times C_3 \times C_7 \times C_{2q+1}$.

当 $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 2)$时,若 $G \cong C_{2^2} \times C_{p_1} \times C_{p_2} \times C_{p_3^2}$,则 $|A(G)| = 2(p_1 - 1)(p_2 - 1)p_3(p_3 - 1) = 2^4p^2q$ 若 $p_3 = q$,则 $(p_1 - 1)(p_2 - 1)(q - 1) = 2^3p^2$,矛盾. 若 $p_3 = p$,则 $(p_1 - 1)(p_2 - 1)(p - 1) = 2^3pq = 2 \cdot 2p \cdot 2q = 2 \cdot 2q \cdot 2p = 2p \cdot 2q \cdot 2$,经计算,当 $2p + 1, 2q + 1$为素数时,有群 $G_{188} \sim G_{189}$,计 2型. 若 $G \cong C_2 \times C_2 \times C_{p_1} \times C_{p_2} \times C_{p_3^2}$,则 $|A(G)| = 2 \cdot 3(p_1 - 1)(p_2 - 1)p_3(p_3 - 1) = 2^4p^2q$ 经计算不可能成立. 当 $(\alpha_1, \alpha_2, \alpha_3)$为其他组合时,经计算群 G均不存在.

(5)当 $k = 4$时,由于 2^4p^2q不能写成 4个互不相同偶数的积,因此,此时群 G不存在.

此外,因 $p \neq q$,所以 G_8 和 G_9 重复 1型, $G_{39} \sim G_{50}$和 $G_{51} \sim G_{60}$至少重复 10型, $G_{77} \sim G_{79}$和 $G_{80} \sim G_{83}$至少重复 3型, $G_{84} \sim G_{89}$和 $G_{90} \sim G_{93}$至少重复 4型, G_{94}, G_{107}和 $G_{95} \sim G_{100}$至少重复 2型, G_{108}和 G_{109}重复 1型, $G_{116} \sim G_{117}$和 $G_{118} \sim G_{119}$重复 2型, $G_{143} \sim G_{156}$和 $G_{157} \sim G_{168}$至少重复 12型, $G_{169} \sim G_{174}$和 $G_{175} \sim G_{178}$至少重复 4型,因此群 G至多有 150型.

参考文献:

[1] Pan J M. *The order of the automorphism group of finite Abelian group*[J]. 云南大学学报:自然科学版, 2003, 26(5): 370-372.

[2] 黄平安,钱国华. 自同构群阶为 prq^2 的限群[J]. 数学学报, 2004, 47(3): 615-620.

[3] Li S R. *On the solution of the eqution* $|Aut(G)| = p^2q^2$ (chinese)[J]. Chinese Science Bulletin, 1995, 40(23): 2124-2127.

[4] 王秀花,黄本文. $|A(G)| = 2^5p^2$ (p为奇素数)的有限 Abel群 G[J]. 武汉大学学报:理学版, 2008, 51(1): 19-24.

[5] 张远达. 有限群构造[M]. 北京:科学出版社, 1982: 80, 137, 138, 412-414.

[6] 俞曙霞. 有限交换 p群的自同构群的阶的几点注记[J]. 数学杂志, 1983(2): 189-194.

[7] 华罗庚. 数论导引[M]. 北京:科学出版社, 1957: 6-7.

[8] 张全超,刘丁酉. 线性群 $GL(n, Z_m)$的换位子群[J]. 湖北民族学院学报:自然科学版, 2008, 26(2): 129-130.

[9] 裴惠生,吴永福. 保持两个等价关系的夹心半群的格林关系和正则性[J]. 信阳师范学院学报:自然科学版, 2008, 21(2): 161-164.

责任编辑:郭红建