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·基础理论研究·

Construction of generalized orthonormal wavelet packets

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Abstract: Orthonormal wavelet packets, which was introduced by Coifman and Meyer in [1], are used to furture decompose wavelet components. In this paper, generalized orthonormal wavelet packets are constructed. The corresponding decomposition and reconstruction algorithms for implementation are also given. The generalized orthonormal wavelet packets we give are more flexible than the wavelet packets in [1].

Key words: multiresolution analysis; wavelet; orthonormal wavelet packets; general orthonormal wavelet packets

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1 Introduction

Orthonormal wavelet packets introduced by Coifman and Meyer (see [1]) are used to furture decompese wavelet components. Orthonormal wavelet packets are extended to non-orthonormal setting (see [2]). This extension is valuable because linear-phase filters cannot be constructed by using compactly supported orthonormal wavelets, but can be constructed by using semi-orthonormal or biorthonormal ones. Wavelet packets with the scaling matrix are studied by Cheng Zhengxing (see [3]). Wavelet packets with the scaling factor a ($a \geq 2$, $a \in Z$) are also investigated (see [4]). In this paper, generalized orthonormal wavelet packets is introduced. The corresponding decomposition and reconstruction algorithms for implementation are also given. The generalized orthonormal wavelet packets we give are more flexible in its application. and It generalizes the result in [1].

Throughout this paper, the space of all square-integrable functions on the real line will be denoted, as usual, by $L^2 = L^2(R)$, the space of square summable sequence will be denoted by l^2, and the notation for inner produce and Fourier transform of function in L^2 is given by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx \tag{1}$$

and

$$\hat{f}(w) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx \tag{2}$$

for any function f, we will always use the notation

$$f_{j,k}(x) = 2^{j/2} f(2^j x - k) \tag{3}$$

2 Orthonormal wavelet and wavelet packets

Let $\psi \in L^2(R)$ be a wavelet function, if $\psi(x)$ satisfy $\int_{-\infty}^{\infty} f(x)\,dx = 0$. An orthonormal bases of wavelet can be constructed from a multiresolution analysis (MRA).

A function $\Phi(x)$ is called a scaling function,

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if $\Phi(x)$ satisfies

$$\Phi(x) = \sum_k p_k \Phi(2x - k), k \in Z \tag{4}$$

If $\Phi(x)$ satisfies

$$\langle \Phi(x-j), \Phi(x-k) \rangle = \delta_{j,k}, j, k \in Z \tag{5}$$

called $\Phi(x)$ an orthonormal scaling function.

We call $\psi(x) \in L^2(R)$ an orthonormal wavelet, if set $\{\psi_{j,k}, j, k \in Z\}$ is an orthonormal bases of $L^2(R)$, i.e. $\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{j,l}\delta_{k,m}, j, k, l, m \in Z$.

Define a subspace $V_j \subset L^2(R)$ by

$$V_j = \mathrm{Clos}_{L^2}\langle \Phi_{j,k}, k \in Z \rangle, j \in Z \tag{6}$$

As usual, $\Phi(x)$ in (4) generates a multiresolution analysis $\{V_j\}_{j\in Z}$ of $L^2(R)$, if $\{V_j\}_{j\in Z}$ defined in (6) satisfy the following properties:

1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots$;
2. $\mathrm{Clos}_{L^2(R)} = (\bigcup_{j\in Z} V_j) = L^2(R)$;
3. $\bigcap_{j\in Z} V_j = \{0\}$;
4. $\forall f(x) \in V_m \Leftrightarrow f(2x) \in V_{m+1}$;
5. $\forall m \in Z$, $V_m = \mathrm{Clos}_{L^2}\{\Phi_{m,n}\}_{n\in Z}$, and set $\{\Phi_{m,n}\}_{n\in Z}$ is an uncondition basis of V_m.

Let $\{W_j\}, j \in Z$ denote the orthonormal complementary subspace of V_j in V_{j+1}, and ψ constitutes a Riesz basis for W_j, i.e.,

$$W_j = \mathrm{Clos}_{L^2}\langle \psi_{j,k} : k \in Z \rangle, j \in Z \tag{7}$$

$$V_{j+1} = V_j \oplus W_j, j \in Z \tag{8}$$

Therefore,

$$L^2(R) = \bigoplus_{j\in Z} W_j \tag{9}$$

From condition (7), it is clear that $\psi(x)$ is in $W_j \subset V_1$, Hence there exists a sequence $\{q_k\}$ such that

$$\psi(x) = \sum_{k\in Z} \Phi(2x - k) \tag{10}$$

In what follows, let us the notation:

$$\begin{cases} \mu_0(x) = \Phi(x) \\ \mu_1(x) = \psi(x) \end{cases} \tag{11}$$

$$\begin{cases} P_0(z) = \frac{1}{2}\sum_k p_k z^k \\ P_1(z) = \frac{1}{2}\sum_k q_k z^k \end{cases} \tag{12}$$

Hence, the two-scale relation of the scaling function Φ and its corresponding wavelet ψ are given by

$$\begin{cases} \mu_0 = \sum_k p_k \mu_0(2x - k) \\ \mu_1 = \sum_k q_k \mu_0(2x - k) \end{cases} \tag{13}$$

or equivalently

$$\begin{cases} \hat{\mu}_0(\omega) = P_0(e^{-i\omega/2})\hat{\mu}_0(\frac{\omega}{2}) \\ \hat{\mu}_1(\omega) = P_1(e^{-i\omega/2})\hat{\mu}_0(\frac{\omega}{2}) \end{cases} \tag{14}$$

This new notation is intended to facilitate the introduction of the following family of functions, called "wavelet packets", these functions give rise to orthonormal bases which can be used to improve the performance of wavelets for time-frequency localization.

Definition 1[1] The functions μ_n, $n = 2l$ or $2l+1$, $l = 0, 1, \ldots$, defined by

$$\begin{cases} \mu_{2l}(x) = \sum_k p_k \mu_l(2x - k) \\ \mu_{2l+1}(x) = \sum_k q_k \mu_l(2x - k) \end{cases} \tag{15}$$

are called "wavelet packets" relative to the orthonormal scaling function $\mu_0 = \Phi$.

3 Construction of generalized orthonormal wavelet packets

Definition 2 A Laurent series is called the Wiener class, which will be denoted by W, if its corresponding sequences $\{a_n\}$ in l_1, namely, $\sum |a_n| < \infty$.

Let us consider an arbitrary Laurent series $R(z)$ of class W, which never vanishes on the unit circle $|z| = 1$. By Wiener's Lemma, we have $\frac{1}{R(z)} \in W$.

Let $\{a_n\} \subset l_1$, we will usuall consider symbols

$$A(z) = \sum_{n\in Z} a_n z^n \tag{16}$$

to select $\{p_k^*\}$ and $\{q_k^*\}$ is in l_1, its corresponding sequences symbols is $P^*(z) = \frac{1}{2}\sum_k p_k^* z^k$, $Q^*(z) = \frac{1}{2}\sum_k q_k^* z^k$. For convenience, let $P_0^* = P^*(z)$, $P_1^*(z) = Q^*(z)$, $P_0(z) = P(z)$, $P_1(z) = Q(z)$. The requirement on the matrix

$$M(z) = \begin{bmatrix} P_0^*(z) & P_0^*(-z) \\ P_1^*(z) & P_1^*(-z) \end{bmatrix} \tag{17}$$

is unitary. i. e. , $M(z) M^*(z) = I$,on $|z| = 1$,where $M^*(z) = \overline{(M(z))}^T$. According to (17) , $M(z)$ is unitary ,then

$$\begin{cases} |P_0^*(z)|^2 + |P_0^*(-z)|^2 = 1 \\ |P_1^*(z)|^2 + |P_1^*(-z)|^2 = 1 \end{cases} \quad (18)$$

$$P_0^*(z)\overline{P_1^*(z)} + P_0^*(-z)\overline{P_1^*(-z)} = 0, |z| = 1 \quad (19)$$

Definition 3 The functions μ_n , $n = 2l$ or $2l + 1$, $l = 0, 1, \ldots$,defined by

$$\begin{cases} \mu_0(x) = \sum_k p_k \mu_0(2x - k) \\ \mu_1(x) = \sum_k q_k \mu_0(2x - k) \\ \mu_{2l}(x) = \sum_k p_k^* \mu_l(2x - k) \\ \mu_{2l+1}(x) = \sum_k q_k^* \mu_l(2x - k) \end{cases} \quad (20)$$

are called "generalized wavelet packets "relative to the orthonormal scaling function $\mu_0 = \Phi$, here , the sequences $\{p_k\}$ and $\{q_k\}$ are two scale sequences.

To describe μ_n , $n \in Z_+$, via its Fourier transform ,we need the dyadic expansion of $n \in Z_+$,namely :

$$n = \sum_{j=1}^{\infty} \varepsilon_j 2^{j-1}, \varepsilon_j \in \{0, 1\} \quad (21)$$

Observe that (21) is always a finite sum and the expansion is unique. if $2^{s_0 - 1} \le n < 2^{s_0}$, $s_0 \in Z_+$,we have $\varepsilon_j = 1$ for $j = s_0 + 1$ and $\varepsilon_j = 0$ for $j > s_0 + 1$. So that $n = \sum_{j=1}^{s_0+1} \varepsilon_j 2^{j-1}$,and $\frac{n}{2} = \frac{\varepsilon_1}{2} + \sum_{j=1}^{s_0} \varepsilon_{j+1} 2^{j-1}$.

As usual ,let [x] denote the largest integer not exceeding x ,and observe that $n = 2[\frac{n}{2}] + \varepsilon_1 = 2n_1 + \varepsilon_1$,here $n_1 = [\frac{n}{2}])$. Similarly , $n_k = [\frac{n_{k-1}}{2}]$,where $n_0 = n$, thus $n_k = \sum_{j=1}^{s_0 - k+1} \varepsilon_{j+k} 2^{j-1}$. For convenience , let $P_{\varepsilon_k}^{(n_k)}(z) = P_{\varepsilon_k}^*(z)$,where $n_k \ge 1$,and $P_{\varepsilon_k}^{(n_k)}(z) = P_{\varepsilon_k}(z)$,where $n_k = 0$.

Theorem 1 Let n be any nonnegative integer and let the dyadic expansion of n be given by (21) , then the Fourier transform of the wavelet packet μ_n is given by

$$\hat{\mu}_n(\omega) = P_{\varepsilon_k}^{(n_k)}(e^{-i\omega/2^k}), \omega \in R \quad (22)$$

Theorem 2 Let Φ be any o. n. scaling function and $\{\mu_n\}$ be given by definition 3 ,then for each $n \in Z_+$,

$$\langle \mu_n(\cdot - j), \mu_n(\cdot - k) \rangle = \delta_{j,k}, j, k \in Z_+ \quad (23)$$

Theorem 3 The family $\{\mu_n\}$ is a general o. n. wavelet packets ,then

$$\langle \mu_{2l}(\cdot - j), \mu_{2l+1}(\cdot - k) \rangle = 0, \quad j, k \in Z, l \in Z_+ \quad (24)$$

The proof method of Theorem 1～Theorem 3 is similar to the proof method of [1 ,Th7. 24～Th7. 26] , therefore ,the proof of Theorem 1～Theorem 3 is ignored.

Let $\{\mu_n\}$ be a family of general wavelet packets corresponding to some o. n. scaling function $\mu_0 = \Phi$. For each $n \in Z_+$,consider the family of subspaces

$$U_j^n = \mathrm{Clos}_{L^2(R)} \langle 2^{j/2} \mu_n(2^j x - k), k \in Z \rangle, j \in Z, n \in Z_+ \quad (25)$$

generated by $\{\mu_n\}$. Recall that

$$\begin{cases} U_j^0 = V_j, j \in Z \\ U_j^1 = W_j, j \in Z \end{cases} \quad (26)$$

where $\{V_j\}$ is MRA of $L^2(R)$ generated by $\mu_0 = \Phi$ and $\{W_j\}$ is the sequence of o. n. complementary (wavelet) subspaces generated by the wavelet $\mu_1 = \psi$, then the orthonormal decomposition $V_{j+1} = V_j \oplus W_j$, $j \in Z$,may be written as

$$U_{j+1}^0 = U_j^0 \oplus U_j^1, j \in Z \quad (27)$$

In the following ,we shall give orthonormal decomposition of $L^2(R)$.

Therorem 4 Let n be any nonnegative integer , then

$$U_{j+1}^n = U_j^{2n} \oplus U_j^{2n+1}, j \in Z \quad (28)$$

According to (20) in definition 3 and Theorem 3 ,it is easy to proof theorem 4.

Theorem 5 For each $j = 1, 2, \cdots$

$$\begin{cases} W_j = U_{j-1}^2 \oplus U_{j-1}^3 \\ W_j = U_{j-2}^4 \oplus U_{j-2}^5 \oplus U_{j-2}^6 \oplus U_{j-2}^7 \\ \cdots\cdots \\ W_j = U_{j-k}^{2^k} \oplus U_{j-k}^{2^k+1} \oplus \cdots \oplus U_{j-k}^{2^{k+1}-1} \\ \cdots\cdots \\ W_j = U_0^{2^j} \oplus U_0^{2^j+1} \oplus \cdots \oplus U_0^{2^{j+1}-1} \end{cases}$$

Furthermore ,for each $m = 0, \cdots, 2^k - 1$, $k = 1, \cdots, j$, and $j = 1, 2, \cdots$,the family

$$\{2^{\frac{j-k}{2}} \mu_{2^k+m}(2^{j-k} x - l) : l \in Z\} \quad (29)$$

is an orthonormal basis of $U_{j-k}^{2^k+m}$

Corollary 1 For each $j = 0, 1, 2, \cdots$,

$$L^2(R) = \underset{j \in Z}{\oplus} W_j$$

$$= \cdots \oplus W_{-1} \oplus W_0 \oplus U_0^2 \oplus U_0^3 \oplus \cdots \quad (30)$$

Of course, the family $\{\psi_{j,k}, \mu_n(\cdot - k): j = \cdots, -1, 0; n = 2, 3, \cdots \text{and } k \in Z\}$ is an o. n. basis of $L^2(R)$.

Let $f_j^n(x) \in U_j^n$, then $f_j^n(x)$ may be written as

$$f_j^n(x) = \sum d_l^{j,n} \mu_n(2^j x - l) \quad (31)$$

By applying (31), $f_{j+1}^n(x)$ can be decomposed into $f_j^{j,n}(x)$, and $f_j^{2n+1}(x)$, then

Decomposition algorithm:

$$d_l^{j,2n+\varepsilon} = \sum_k a_{k-2l}^{(n)} d_k^{j+1,n}, \varepsilon = 0, 1,$$

Reconstruction algorithm:

$$d_l^{j+1,n} = \sum_k [p_{l-2k}^{(n)} d_l^{j,2n} + q_{l-2k}^{(n)} d_l^{j,2n+1}],$$

Remark $p_{l-2k}^0 = p_{l-2k}$, $q_{l-2k}^{(0)} = q_{l-2k}$, $p_{l-2k}^{(n)} = p_{l-2k}^*$, $q_{l-2k}^{(n)} = q_{l-2k}^*$ $(n \geq 1)$.

4 Conclusion

In this paper, to select $\{p_k^*\}$ and $\{q_k^*\}$ is in l_1 and $M(z)$ defined in (17) is unitary. Obviously, reconstruction sequences $\{p_k\}$ and $\{q_k\}$ of and arbitray scaling function and its correponding wavelet satisfy (17). Of course, sequences satisfing condition (17) isn't sure reconstruction sequences (see [2]). Hence, Generalized orthonormal wavelet packets can reduce to orthonormal wavelet packets introduced by Cofiman and Meyer[1] when $\{p_k^*\} = \{p_k\}$ and $\{q_k^*\} = \{q_k\}$.

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广义的正交小波包

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摘 要: COIFMANT和MEYER在[1]中引入了正交小波包的概念, 它是为了进一步分解小波的分量. 本文在此基础上引入了广义小波包的概念并给出相应的分解与重构算法. 它推广了文[1]的结果, 且这种小波包比传统的正交小波包在应用上具有较大的灵活性.

关键词: 多分辨分析; 小波; 正交小波包; 广义的正交小波包

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