DOI: 10.3969/j.issn.1003-0972.2011.04.001

Studies on a Delayed SIR Epidemic Model with Saturated Rate

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Abstract: A delayed SIR epidemic model with saturated rate is introduced. The local stabilities of its equilibria as well as the effects of delay on the reproduction number of the model are studied by constructing Lyapunov function.

Key words: SIR epidemic model; delay; equilibrium; stability; Lyapunov function.

CLC number: O175.1　　**Article character**: A　　**Article ID**: 1003-0972(2011)04-0421-04

一类带有饱和感染率的时滞 SIR 传染病模型研究

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摘　要: 介绍一类带有饱和感染率的时滞 SIR 传染病模型，通过构造 Lyapunov 函数，研究了该模型平衡点的局部稳定性以及时滞对基本再生数的影响.

关键词: SIR 传染病模型; 时滞; 平衡点; 稳定性; Lyapunov 函数

中图分类号: O175.1　　**文献标志码**: A　　**文章编号**: 1003-0972(2011)04-0421-04

0　Introduction

Epidemiological models have been used to study ecological and epidemiological phenomena extensively[1-8]. Let $S(t)$, $I(t)$ and $R(t)$ be the respective population densities of susceptible, infective and recovered. We assume that $S(t) + I(t) + R(t) = N(t)$, a nonconstant population with carrying capacity $K(N(t) \leqslant K)$. These variables will be normalized to $N(t) = 1$, because most models assumed the population in question is constant. In [4], a standard SIR epidemic model takes the form

$$\begin{cases} \dot{S}(t) = \mu - \mu S(t) - \beta S(t) I(t), \\ \dot{I}(t) = \beta S(t) I(t) - (\mu + \gamma) I(t), \\ \dot{R}(t) = \gamma I(t) - \mu R(t), \end{cases} \tag{1}$$

where parameters μ, β, γ are positive constants in which μ is the death rate and recruitment rate of some populations. All newborns and immigrants are susceptible.

Based on the Hethcote's model, we introduce our model through the incorporation of variable population, disease-induced mortality and time delay of the infective class into the classic model in [4]. Thus, we proposed the following model in which d, the renewal or recruitment of individuals through birth and/or immigration, is different from the death rate μ. So, we investigated the model with the force of infection at any time t given by

$$\beta e^{-\mu_1 \tau} \frac{S(t) I(t-\tau)}{1 + \alpha I(t-\tau)},$$

and the model has the following form:

$$\begin{cases} \dot{S}(t) = d - \mu S(t) - \beta e^{-\mu_1 \tau} \dfrac{S(t) I(t-\tau)}{1 + \alpha I(t-\tau)}, \\ \dot{I}(t) = \beta e^{-\mu_1 \tau} \dfrac{S(t) I(t-\tau)}{1 + \alpha I(t-\tau)} - (\mu + \gamma) I(t), \\ \dot{R}(t) = \gamma I(t) - \mu R(t), \end{cases} \tag{2}$$

Received date: 2010-12-28; **Revised date**: 2011-04-06; * . **Corresponding author, E-mail**: xywangxia@163.com

Foundation item: The National Natural Science Foundation of China(10771179); the Scientific and Technological Project of Henan Province (092102210070); the Youth Science Foundations of Xinyang Normal University(20100070); the Young Backbone Teacher Foundation of the Education Department of Henan Province(2010GGJS-104) and Xinyang Normal University

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where β denotes the per capita effective daily contact rate, μ_1 denotes the disease-induced death rate in the interval $(0,\tau)$, μ is the death rate of the population ($\mu_1 \ll \mu$), γ denotes the daily recovery rate of the infectious, τ denotes the time delay of a full blown infection to take place and parameters $d,\mu,\mu_1,\beta,\gamma,\tau$ are positive constants.

This means that infection is caused by infectious group, infected τ units of time earlier. However, not all those infected will survive after τ units of time, which suggested the introduction of the survival term $e^{-\mu_1\tau}$. This also implies that there is an implicity define class, but for the sake of simplicity and elucidation, we assume that the susceptible and exposed groups are indistinguishable.

1 Boundedness and equilibrium

Now we show that all solutions of system (2) are bounded by the idea of dissipativity.

Theorem 1 System (2) is dissipative.

Proof. Let $(S(t),I(t),R(t))$ be any solution with non-negative initial condition $(S(0),I(0),R(0))$. Since $\dot{S}(t) \leqslant d-\mu S(t)$, then

$$\limsup_{t\to\infty} S(t) \leqslant M,$$

where $M=\{S(0),K\}$, $K<\infty$ being the carrying capacity. Now we consider the function

$$X(t)=S(t)+I(t)+R(t),$$

then

$$\dot{X}(t)=d-\mu(S(t)+I(t)+R(t))=d-\mu X(t).$$

By applying the theorem of Birkhoff and Rota[6] on differential inequalities, we get

$$0 \leqslant X(t) \leqslant \frac{d}{\mu}+X(S(0),I(0),R(0))e^{-\mu t}.$$

Thus, for $t\to\infty$, $0\leqslant X(t) \leqslant \frac{d}{\mu}$. Therefore, all the solutions of system (2) enter the region $\Gamma=\{(S,I,R)\in R^3_+ \mid X\leqslant \frac{d}{\mu}+\varepsilon \text{ for any } \varepsilon>0\}$, a compact invariant set of the non-negative cone R^3_+, which is attracting. This completes the proof.

Γ is also an asymptotic global attractor for all solutions of system (2). Thus, it makes sense to study the dynamics of system (2) in Γ rather than in R^3_+.

For any $t\in R$, we consider any pair of variables from the set $\{S,I,R\}$. Let us consider $u(t)=(S(t),I(t))\in\Omega$, where

$$\Omega=\{(S,I)\in R^2_+ \mid S+I\leqslant \frac{d}{\mu}\},$$

or $u(t)=(I(t),R(t))\in\Omega_1$, where

$$\Omega_1=\{(I,R)\in R^2_+ \mid I+R\leqslant \frac{d}{\mu}\}.$$

For $d=\mu, N\leqslant 1$ in this case (that is, $I+R\leqslant 1$ or $S+I\leqslant 1$).

Denote by $C((-\infty,0],R^2)$ the Banach space of continuous functions mapping the interval $(-\infty,0]$ into R^2, with the topology of uniform convergence[6]. That is, for $\varphi\in C((-\infty,0],R^2)$, the norm of φ is defined as

$$\|\varphi\|=\sup_{\theta\in(-\infty,0]}|\varphi(\theta)|,$$

where $|\cdot|$ represents the usual norm in R^2. Furthermore, for $x\in C((-\infty,0],R^2)$ and $t\in[0,\tau]$, we define $u_t\in C$ as

$$u_t(\theta)=u(t+\theta)=(S(t+\theta),I(t+\theta)),$$

or $u_t(\theta)=u(t+\theta)=(I(t+\theta),R(t+\theta))$ for $\theta\in(-\infty,0]$.

System (2) can be written as an autonomous system of delay differential equations:

$$\dot{u}(t)=f(u_t), \tag{3}$$

where $R(t)$ is given by $R(t)=1-(S(t)+I(t))$, provided $d=\mu$ for any $t\in[0,\tau]$.

Next, we only investigate the following subsystem

$$\begin{cases}\dot{S}(t)=d-\mu S(t)-\beta e^{-\mu_1\tau}\dfrac{S(t)I(t-\tau)}{1+\alpha I(t-\tau)},\\ \dot{I}(t)=\beta e^{-\mu_1\tau}\dfrac{S(t)I(t-\tau)}{1+\alpha I(t-\tau)}-(\mu+\gamma)I(t).\end{cases} \tag{4}$$

From (4), we can easily get the disease-free equilibrium

$$E_0=(S_0,0)=(\frac{d}{\mu},0)$$

and the endemic equilibrium $E^*=(S^*,I^*)$, where

$$S^*=\frac{\mu+\gamma+\alpha d}{\alpha\mu+\beta e^{-\mu_1\tau}},$$

$$I^*=\frac{d\beta e^{-\mu_1\tau}-\mu(\mu+\gamma)}{(\mu+\gamma)(\alpha\mu+\beta e^{-\mu_1\tau})}.$$

Let

$$\tau^*=\frac{1}{\mu_1}\ln\frac{d\beta}{\mu(\mu+\gamma)},$$

then it is obtained that if $\tau=\tau^*$, then $I^*=0$, $S^*=\frac{d}{\mu}$ and E^* becomes coincident with E_0; if $\tau>\tau^*$, there only exists E_0; if $\tau<\tau^*$, then E^* exists, but in the sequel, we shall assume that system (2) is scaled such that $S+I+R=1$ (with $d=\mu$). That is,

$$s(t)=\frac{S(t)}{N}, i(t)=\frac{I(t)}{N}, r(t)=\frac{R(t)}{N}.$$

In this case, only the first two equations of system (2) are relevant in the analysis. Without any ambiguity, we have use the same notations, with the understanding that now $S+I+R=1$.

By Corollary 5.2 in Ref. [6], we have the following Lemma.

Lemma 1 Assume that $\omega_1(\cdot)$ and $\omega_2(\cdot)$ are non-negative continuous scalar functions from R_+ to R_+, such that

$$\omega_1(0) = 0 = \omega_2(0), \lim_{t\to\infty} \omega_1(t) = +\infty,$$

and $V: C \to \mathrm{R}$ is continuous and satisfies

$$V(\varphi) \geqslant \omega_1(|\varphi(0)|),$$

$$\dot{V}(\varphi)|_{(3)} \leqslant -\omega_1(|\varphi(0)|). \quad (5)$$

Then, the solution $u = 0$ of equation (3) is uniformly stable and every solution is bounded.

2 Stability of the equilibria

In this section, we discuss the local stability of the disease-free equilibrium $E_0 = (S_0, 0)$ and the endemic equilibrium $E^* = (S^*, I^*)$ of system (4). It is more convenient to choose the variable (I, R) instead of (S, I) which helps us to obtain an expression for the time delay τ, and consider the following subsystem of (2):

$$\begin{cases} \dot{I}(t) = \beta e^{-\mu_1\tau} \dfrac{S(t)I(t-\tau)}{1+\alpha I(t-\tau)} - (\mu+\gamma)I(t), \\ \dot{R}(t) = \gamma I(t) - \mu R(t), \end{cases} \quad (6)$$

where $S(t) \equiv 1$, and equation (6) becomes

$$\begin{cases} \dot{I}(t) = \beta e^{-\mu_1\tau} \dfrac{I(t-\tau)}{1+\alpha I(t-\tau)} - (\mu+\gamma)I(t), \\ \dot{R}(t) = \gamma I(t) - \mu R(t). \end{cases} \quad (7)$$

Then the characteristic equation of system (7) at the disease-free equilibrium $E'_0 = (I_0, R_0) = (0, 0)$ (here $S_0 + I_0 + R_0 = 1, S_0 = 1, I_0 = 0, R_0 = 0$) takes the form

$$(\lambda+\mu)(\lambda+\mu+\gamma-\beta e^{-\mu_1\tau}e^{-\lambda\tau}) = 0. \quad (8)$$

Therefore, one of the characteristic roots is $\lambda = -\mu < 0$. Other roots of (8) are determined by the following equation

$$\lambda+\mu+\gamma-\beta e^{-\mu_1\tau}e^{-\lambda\tau} = 0. \quad (9)$$

When $\lambda = 0$, from equation (9), the delay-induced reproduction number R_1 given below is obtained. The disease-free equilibrium being the most important from the eco-biomedical point of view, parameters that drive the disease dynamics are paramount in determing the stability of this infection-free steady state. A condition for the local stability of the disease-free equilibrium E'_0 of system (7) is that the delay-induced reproduction number

$$R_1 = \frac{\beta e^{-\mu_1\tau}}{\mu+\gamma} = R^0 e^{-\mu_1\tau} < 1, \quad (10)$$

where

$$R^0 = \frac{\beta}{\mu+\gamma}$$

is called the basic reproduction number (BRN). Even though the introduction of time delays is harmless[7] to the persistence of the disease and the existence of positive equilibrium at least from the mathematical point of view, it is a destabilizing process in the sense that increasing the time delay could cause a stable equilibrium to become unstable or cause the population to fluctuate[8]. For example, if $\tau \to \infty$, then $R_1 \to 0$ or τ is large (the disease is almost eradicated, which is not realistic, since diseases such as HIV have a very long incubation period) while for $\tau \to 0, R_1 \to R_0$. Thus, it is important to find the critical value of τ for which the disease will not spread. We have found that the endemic equilibrium exists if and only if

$$\tau < \tau^* = \frac{1}{\mu_1}\ln\frac{d\beta}{\mu(\mu+\gamma)}.$$

Let $R_1 = 1$, solving for τ, we obtain the critical time delay as

$$\tau_c = \frac{1}{\mu_1}\ln\frac{\beta}{\mu+\gamma}.$$

Taking equality in the expression for τ^*, it can be shown that $\tau^* - \tau_c > 0$. Thus, $0 \leqslant \tau_c < \tau^*$.

If infective individuals (in the latent phase of the disease) or those who come into contact with infectious individuals are detected and quarantine (during this time interval), then the disease will likely die down if $R_0 < e^{\mu_1\tau}$.

Theorem 2 The disease-free equilibrium E'_0 of system (7) is locally asymptotically stable when $R_1 < 1$.

Proof. As $\varphi(\theta)$ for any $\theta \in (-\infty, 0]$ and $\varphi(0) > 0$, $I(t) > 0$ for all $t \geqslant 0$, where $I(t)$ is any solution of

$$\dot{I}(t) = \beta e^{-\mu_1\tau} \frac{I(t-\tau)}{1+\alpha I(t-\tau)} - (\mu+\gamma)I(t). \quad (11)$$

Therefore, it is enough to consider the Lyapunov functional

$$V(I(t)) = I(t) + \beta e^{-\mu_1\tau}\int_{t-\tau}^{t}\frac{I(v)}{1+\alpha I(v)}dv, \quad (12)$$

which satisfies $V(I(t)) \geqslant I(t) = |I(t)|$ for any $t \geqslant 0$. So, we have

$$\begin{aligned} \dot{V}(I(t))|_{(11)} = {} & \dot{I}(t) + \\ & \beta e^{-\mu_1\tau}\frac{d}{dt}\int_{t-\tau}^{t}\frac{I(v)}{1+\alpha I(v)}dv = \\ & \dot{I}(t) + \beta e^{-\mu_1\tau}\left(\frac{I(t)}{1+\alpha I(t)} - \frac{I(t-\tau)}{1+\alpha I(t-\tau)}\right) = \\ & \beta e^{-\mu_1\tau}\frac{I(t)}{1+\alpha I(t)} - (\mu+\gamma)I(t) \leqslant \\ & \beta e^{-\mu_1\tau}I(t) - (\mu+\gamma)I(t), \end{aligned} \quad (13)$$

$$\dot{V}(I(t)) \leqslant (\beta e^{-\mu_1\tau} - (\mu+\gamma))I(t) < 0, R_1 < 1.$$

Therefore, if $R_1 < 1$, then $E'_0 = (0, 0)$ is asymptotically stable.

Now let us consider the local asymptotic stability of the endemic equilibrium $E^* = (S^*, I^*)$ of system (4).

Theorem 3 Whenever the endemic equilibrium E^* of system (4) exists, it is locally asymptotically stable.

Proof. Denote $W_1 = S - S^*, W_2 = I - I^*$, then the linear system of system (4) as follows:

$$\begin{cases}\dot{W}_1(t) = -\left(\beta e^{-\mu_1\tau}\dfrac{I^*}{1+\alpha I^*}+\mu\right)W_1(t) - \\ \qquad \beta e^{-\mu_1\tau}\dfrac{S^*}{(1+\alpha I^*)^2}W_2(t-\tau), \\ \dot{W}_2(t) = \beta e^{-\mu_1\tau}\dfrac{I^*}{1+\alpha I^*}W_1(t) - (\mu+\gamma)W_2(t) + \\ \qquad \beta e^{-\mu_1\tau}\dfrac{S^*}{(1+\alpha I^*)^2}W_2(t-\tau).\end{cases} \tag{14}$$

Now, we consider the Lyapunov functional

$$V(W(t)) = \frac{1}{2}W_2^2(t) + \frac{1}{2}p(W_1(t)+W_2(t))^2 + \frac{1}{2}\beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}\int_{t-\tau}^{t}W_2^2(\theta)\,d\theta, \tag{15}$$

where $p > 0$ is a constant. We observe that

$$V(W(t)) \geqslant \omega_1(|W(t)|) = \frac{1}{2}W_2^2(t) + \frac{1}{2}p(W_1(t)+W_2(t))^2, \tag{16}$$

where ω_1 is a positive definite quadratic form of W_1 and W_2, since $p > 0$. Hence, $\omega_1 \geqslant 0$, $\omega_1 = 0$ if and only if $|W(t)| = 0$, and

$$\lim_{|W(t)|\to+\infty}\omega_1(|W(t)|) = +\infty.$$

The time derivative of $V(W(t))$ along the solution of system (14) is given by

$$\begin{aligned}\dot{V}(W(t))|_{(14)} &= W_2(t)\dot{W}_2(t) + p(W_1(t)\dot{W}_1(t) + W_1(t)\dot{W}_2(t) + W_2(t)\dot{W}_1(t) + W_2(t)\dot{W}_2(t)) + \\ &\quad \frac{1}{2}\beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}\frac{d}{dt}\int_{t-\tau}^{t}W_2^2(\theta)\,d\theta = \\ &\quad W_2(t)\Big(\beta e^{-\mu_1\tau}\frac{I^*}{1+\alpha I^*}W_1(t) - (\mu+\gamma)W_2(t) + \beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_2(t-\tau)\Big) + \\ &\quad p\Big(W_1(t)\Big(-\Big(\beta e^{-\mu_1\tau}\frac{I^*}{1+\alpha I^*}+\mu\Big)W_1(t) - \beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_2(t-\tau)\Big) + \\ &\quad \beta e^{-\mu_1\tau}\frac{I^*}{1+\alpha I^*}W_1^2(t) - (\mu+\gamma)W_1(t)W_2(t) + \\ &\quad \beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_1(t)W_2(t-\tau) - \Big(\beta e^{-\mu_1\tau}\frac{I^*}{1+\alpha I^*}+\mu\Big)W_1(t)W_2(t) - \\ &\quad \beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_2(t)W_1(t-\tau) + \beta e^{-\mu_1\tau}\frac{I^*}{1+\alpha I^*}W_1(t)W_2(t) - \\ &\quad (\mu+\gamma)W_2^2(t) + \beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_2(t)W_2(t-\tau)\Big) + \\ &\quad \frac{1}{2}\beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}(W_2^2(t) - W_2^2(t-\tau)).\end{aligned}$$

After some algebraic manipulations, we obtain

$$\begin{aligned}\dot{V}(W(t))|_{(14)} &= -(\mu+\gamma)W_2^2(t) + \beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_2(t)W_2(t-\tau) - \\ &\quad p\Big(\beta e^{-\mu_1\tau}\frac{I^*}{1+\alpha I^*}+\mu\Big)W_1^2(t) + \beta e^{-\mu_1\tau}\frac{I^*}{1+\alpha I^*}pW_1^2(t) + \\ &\quad \Big(\beta e^{-\mu_1\tau}\frac{I^*}{1+\alpha I^*} - p(2\mu+\gamma)\Big)W_1(t)W_2(t) - p(\mu+\gamma)W_2^2(t) + \\ &\quad \frac{1}{2}\beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_2^2(t) - \frac{1}{2}\beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_2^2(t-\tau).\end{aligned} \tag{17}$$

Choosing $\beta e^{-\mu_1\tau}\dfrac{I^*}{1+\alpha I^*} = p(2\mu+\gamma)$, and

$$\beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_2(t)W_2(t-\tau) \leqslant \frac{1}{2}\beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}(W_2^2(t) + W_2^2(t-\tau)),$$

where $\beta e^{-\mu_1\tau}\dfrac{S^*}{1+\alpha I^*} = \mu+\gamma$, then equation (17) becomes

$$\begin{aligned}\dot{V}(W(t))|_{(14)} &= -(\mu+\gamma)W_2^2(t) - \beta e^{-\mu_1\tau}\frac{I^*}{1+\alpha I^*}pW_1^2(t) - p\mu W_1^2(t) + \\ &\quad \beta e^{-\mu_1\tau}\frac{I^*}{1+\alpha I^*}pW_1^2(t) - (\mu+\gamma)pW_2^2(t) + \beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_2^2(t) = \\ &\quad -(\mu+\gamma)W_2^2(t) - p\mu W_1^2(t) - (\mu+\gamma)pW_2^2(t) + \beta e^{-\mu_1\tau}\frac{S^*}{(1+\alpha I^*)^2}W_2^2(t) = \\ &\quad -(\mu+\gamma)(1+p)W_2^2(t) - p\mu W_1^2(t) + \frac{\mu+\gamma}{1+\alpha I^*}W_2^2(t) \leqslant \\ &\quad -(\mu+\gamma)(1+p)W_2^2(t) - p\mu W_1^2(t) + (\mu+\gamma)W_2^2(t) = \\ &\quad -(\mu+\gamma)pW_2^2(t) - \mu pW_1^2(t) \leqslant \\ &\quad -\mu p(W_1^2(t) + W_2^2(t)) = \\ &\quad -\omega_2(|W(t)|).\end{aligned} \tag{18}$$

From Lemma 1, this completes the proof.

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$$e'_\varepsilon(t) \leqslant -\varepsilon e^{\frac{p+1}{2}}(t). \tag{41}$$

另外，在式(40)中，当$\varepsilon < \frac{1}{1+\gamma}e^{\frac{p-1}{2}}(0)$时，存在$\theta$使得$0 < \theta \leqslant 1-(1+\gamma)e^{\frac{p-1}{2}}(0)$，那么

$$\theta e(t) \leqslant e_\varepsilon(t) \leqslant (2-\theta)e(t). \tag{42}$$

由式(41)和(42)知

$$e'_\varepsilon(t) \leqslant -\frac{\varepsilon}{(2-\theta)^{-\frac{p+1}{2}}}e_\varepsilon^{\frac{p+1}{2}}(t),$$

因此

$$\left(e_\varepsilon^{-\frac{p-1}{2}}(t)\right)' \geqslant \frac{(p-1)\varepsilon}{2(2-\theta)^{-\frac{p+1}{2}}}. \tag{43}$$

在$[0,t]$上积分式(43)可得

$$e_\varepsilon^{-\frac{p-1}{2}}(t) \geqslant e_\varepsilon^{\frac{p-1}{2}}(0) + \frac{(p-1)\varepsilon}{2(2-\theta)^{-\frac{p+1}{2}}}t,$$

则存在$C>0$，使得

$$e(t) \leqslant \frac{1}{\theta}e_\varepsilon(t) \leqslant C(1+t)^{-\frac{2}{p-1}},$$

因此$E(t) \leqslant C(1+t)^{-\frac{2}{p-1}}$.

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