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一类具有 Lévy 噪音的随机竞争系统的最优收获策略

方 壮*,李祖雄

(湖北民族学院 理学院,湖北 恩施 445000)

摘 要:研究了一类具有 Lévy 噪音随机竞争系统的最优收获问题.通过随机分析方法证明了相应解的随机一致有界性,进而针对给定的优化目标泛函,利用变分方法和对偶原理得到了收获策略的一个必要条件.

关键词:Lévy 噪音;随机竞争系统;最优控制

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Optimal Harvesting Strategies for a Stochastic Competitive System with Lévy Noises

FANG Zhuang*, LI Zuxiong

(School of Science, Hubei University for Nationalities, Enshi 445000, China)

Abstract: Optimal harvesting problems of a stochastic competitive system with Lévy noises were investigated. Stochastic uniform boundedness of the solution of corresponding system was proved. For the given cost functional, a necessary condition of the optimal harvesting strategies was established by employing variational methods and dual principle.

Key words: Lévy noises; stochastic competitive system; optimal harvesting strategies

0 引言

随着人类社会的不断发展,环境的破坏和生态资源的过度开采已经导致人类生存环境的恶化和许多种群的灭绝.幸运的是,生态系统的良性循环和生态资源的有效管理和利用,已成为人类共同关注的问题.越来越多的生态学家和数学工作者开始进行定性甚至定量的研究,其中各种各样的数学模型被提出并得到深入研究,得到了许多精彩而有效的结果.早期这方面的研究始于对于确定单种群模型最优收获策略的研究[1,2],接着相互作用的种群系统的最优收获问题被广泛研究[3-5].然而,现实中,生态系统的演化不可避免地受到各种环境因素的干扰,因此为了更准确地刻画真实的生态系统,随机微分方程成了学者们首选的数学工具.近几年来,由随机微分方程刻画的各种生态系统被广泛而深入地研究[2].然而,现有的绝大部分研究都是基于白噪声的随机种群系统.众所周知,白噪声描述的是各种温和的微小噪音的叠加,而对于剧烈的变化尤其是环境的突变对生态系统的扰动,如山洪爆发、地震灾害、瘟疫等,Lévy 噪音的刻画更贴近现实.为此,由 Lévy 噪音刻画的生态系统开始受到学者们的关注[6-8].刘蒙等[7] 研究了如下模型的渐近行为.

$$\begin{cases} \mathrm{d}x_1 = x_1(r_1 - a_{11}x_1 - a_{12}x_2)\mathrm{d}t + \\ \qquad \sigma_1 x_1 \mathrm{d}B_1 + \int_B c_1(u)x_1(t^-)\tilde{N}(\mathrm{d}t,\mathrm{d}u), \\ \mathrm{d}x_2 = x_2(r_2 - a_{21}x_1 - a_{22}x_2)\mathrm{d}t + \\ \qquad \sigma_2 x_2 \mathrm{d}B_2 + \int_B c_2(u)x_2(t^-)\tilde{N}(\mathrm{d}t,\mathrm{d}u), \\ (x_1(0),x_2(0)) \in \mathbf{R}_+^2, \end{cases}$$

其中:x_i 表示种群密度;正常数 $r_i(i=1,2)$ 代表种群的内禀增长率;$c_i(u)$ 为定义在 B 上的正函数;$B_1(t)$,$B_2(t)$ 为实值标准布朗运动.$N(t)$ 是一个泊松计数测度,它在集合 B 上具有特征测度 $\lambda(\cdot)$,$\tilde{N}(\mathrm{d}t,\mathrm{d}u) = N(\mathrm{d}t,\mathrm{d}u) - \lambda(\mathrm{d}u)\mathrm{d}t$,并且始终假设 $B_1(t)$,$B_2(t)$ 及 $N(t)$ 相互独立.

1 最优收获问题的提出

考虑如下收获系统:

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作者简介:方壮(1980—),男,湖北大悟人,讲师,博士,主要从事数字图像处理及数学建模研究.

$$\begin{cases} dx_1 = x_1(r_1 - h_1(t) - a_{11}x_1 - a_{12}x_2)dt + \\ \qquad \sigma_1 x_1 dB_1 + \int_B c_1(u)x_1(t^-)\tilde{N}(dt,du), \\ dx_2 = x_2(r_2 - h_2(t) - a_{21}x_1 - a_{22}x_2)dt + \\ \qquad \sigma_2 x_2 dB_2 + \int_B c_2(u)x_2(t^-)\tilde{N}(dt,du), \\ x_1(0^-) = x_1, x_2(0^-) = x_2, \end{cases} \tag{1}$$

这里,$h(t)=(h_1(t),h_2(t))$ 为收获量.进一步,设优化目标泛函为:

$$J(h(t)) = E\left[\int_0^T e^{-\lambda t}(x_2 h_1(t) + x_2 h_2(t))dt\right], \tag{2}$$

并设 Λ 为容许收获策略集.本文的目标是在集合 Λ 上最小化泛函 $J(h(t))$,即寻求一个 $\bar{h} = (\bar{h}_1, \bar{h}_2) \in \Lambda$,使得

$$J(\bar{h}) = \inf_{h \in \Lambda} J(h). \tag{3}$$

2 解的一致有界性

引理1 设$(x_1(t),x_2(t))$是系统(1)的解,则存在正常数 M,使得

$$\sup_{t\in[0,T]} E[x_1^p(t) + x_2^p(t)] \leqslant M.$$

证明 由伊藤公式有

$$\begin{aligned} de^t(x_1^p + x_2^p) = e^t\{ & -a_{11}px_1^{p+1} - a_{12}px_1^p x_2 + \\ & x_1^p[1 + pr_1 - p\int_B c_1(u)\lambda(du) + \\ & \frac{\sigma_1^2}{2}p(p-1) + \int_B((1+c_1(u))^p - 1)\lambda(du)] - \\ & a_{22}px_2^{p+1} - a_{21}px_1x_2^p + x_2^p[1 + pr_2 - \\ & p\int_B c_2(u)\lambda(du) + \frac{\sigma_2^p}{2}p(p-1) + \\ & \int_B((1+c_1(u))^p - 1)\lambda(du)]\}dt + \\ & e^t[px_1^p\sigma_1 dB_1 + px_2^p\sigma_2 dB_2] + \\ & e^t\int_B[x_1^p((1+c_1(u))^p - 1) + \\ & x_2^p((1+c_2(u))^p - 1)]\tilde{N}(dt,du). \end{aligned}$$

上式两边取期望即得:

$$E[e^t(x_1^p + x_2^p)] = e^t(x_1^p(0) + x_2^p(0)) + E\left[\int_0^t e^s G(s)ds\right].$$

注意到 $G(s) \leqslant \bar{M}$,有

$$E[e^t(x_1^p + x_2^p)] \leqslant e^t(x_1^p(0) + x_2^p(0)) + \bar{M}e^t.$$

从而有

$$E[x_1^p + x_2^p] \leqslant x_1^p(0) + x_2^p(0) + \bar{M},$$

故有 $\sup\limits_{t\in[0,T]} E(x_1^p(t) + x_2^p(t)) \leqslant M$.证毕.

类似地,可以得到如下引理.

引理2 设$(x_1(t),x_2(t))$是系统(1)的解,则存在正常数 K,使得

$$\sup_{t\in[0,T]} E[x_1(t)x_2(t)] \leqslant K.$$

3 最优收获策略的必要条件

接下来,利用变分方法和对偶原理建立最优收获策略的必要条件.

引理3 设$(\bar{h}_1,\bar{h}_2)$为系统(1)的最优收获策略,对任意正数 ε 及 $\Delta h_i \in \Lambda$,进一步假设 $h_i^\varepsilon = \bar{h}_i + \varepsilon\triangle h_i \in \Lambda$.设 x_i^ε 为系统(1)对应于 h_i^ε 的解,则有 $x^\varepsilon \to \bar{x}$.

证明 记

$$z_i^\varepsilon = \frac{x_i^\varepsilon - \bar{x}_i}{\varepsilon}, i = 1,2,$$

则系统(1)可化为

$$\begin{cases} dz_1^\varepsilon = (r_1 z_1^\varepsilon - a_{11}(x_1^\varepsilon + \bar{x}_1)z_1^\varepsilon - a_{12}x_2^\varepsilon z_1^\varepsilon - \\ \qquad a_{12}\bar{x}_1 z_2^\varepsilon - h_1^\varepsilon z_1^\varepsilon - \bar{x}_1\triangle h_1)dt + \\ \qquad \sigma_1 z_1^\varepsilon dB_1 + \int_B c_1(u)z_1^\varepsilon\tilde{N}(dt,du), \\ dz_2^\varepsilon = (r_2 z_2^\varepsilon - a_{21}x_2^\varepsilon z_1^\varepsilon - a_{21}\bar{x}_1 z_2^\varepsilon - \\ \qquad a_{22}(x_2^\varepsilon + \bar{x}_2)z_2^\varepsilon - h_2^\varepsilon z_2^\varepsilon - \bar{x}_2\Delta h_2)dt + \\ \qquad \sigma_2 z_2^\varepsilon dB_2 + \int_B c_2(u)z_2^\varepsilon\tilde{N}(dt,du), \\ z_1^\varepsilon(0) = z_2^\varepsilon(0) = 0. \end{cases} \tag{4}$$

记

$$dz^\varepsilon = \boldsymbol{A}z^\varepsilon dt + \boldsymbol{A}'dt + \boldsymbol{C}dB + \int_B \boldsymbol{D}\tilde{N}(dt,du), \tag{6}$$

其中

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \boldsymbol{A}' = \begin{pmatrix} -\bar{x}_1\Delta h_1 \\ -\bar{x}_2\Delta h_2 \end{pmatrix} = -\bar{\boldsymbol{x}}\Delta\boldsymbol{h}^{\mathrm{T}},$$

$$\boldsymbol{C} = \begin{pmatrix} \sigma_1 z_1^\varepsilon \\ \sigma_2 z_2^\varepsilon \end{pmatrix}, \boldsymbol{D} = \begin{pmatrix} c_1(u)z_1^\varepsilon \\ c_2(u)z_2^\varepsilon \end{pmatrix},$$

$$A_{11} = r_1 - a_{11}(x_1^\varepsilon + \bar{x}_1) - a_{12}x_2^\varepsilon - h_1^\varepsilon,$$

$$A_{12} = -a_{12}\bar{x}_1, A_{21} = -a_{21}x_2^\varepsilon,$$

$$A_{22} = r_2 - a_{21}\bar{x}_1 - a_{22}(x_2^\varepsilon + \bar{x}_2) - h_2^\varepsilon.$$

注意到 $\sup\limits_{t\in[0,T]} E[|\boldsymbol{A}|^2] \leqslant M_1$,从而有

$$z^\varepsilon = \int_0^t \boldsymbol{A}z^\varepsilon dt + \int_0^t \boldsymbol{A}'dt + \int_0^t \boldsymbol{C}dB +$$

$$\int_0^t \int_B \boldsymbol{D}\tilde{N}(\mathrm{d}t,\mathrm{d}u),$$

$$\|z^\varepsilon\|^2 \leqslant 4\left\|\int_0^t \boldsymbol{A}z^\varepsilon \mathrm{d}t\right\|^2 + 4\left\|\int_0^t \boldsymbol{A}'\mathrm{d}t\right\|^2 + 4\left\|\int_0^t \boldsymbol{C}\mathrm{d}B\right\|^2 + 4\left\|\int_0^t \boldsymbol{D}\tilde{N}(\mathrm{d}t,\mathrm{d}u)\right\|^2,$$

$$E[\|z^\varepsilon\|^2] \leqslant 4E\left[\left\|\int_0^t \boldsymbol{A}z^\varepsilon \mathrm{d}t\right\|^2\right] + 4E\left[\left\|\int_0^t \boldsymbol{A}'\mathrm{d}t\right\|^2\right] + 4E\left[\left\|\int_0^t \boldsymbol{C}\mathrm{d}B\right\|^2\right] + 4E\left[\left\|\int_0^t \boldsymbol{D}\tilde{N}(\mathrm{d}t,\mathrm{d}u)\right\|^2\right] \leqslant 4M_1\int_0^t E[\|z^\varepsilon\|^2]\mathrm{d}s + L(t),$$

即有

$$\sup_{t\in[0,T]} E[\|z^\varepsilon\|] \leqslant E\left[\int_0^T L'(s)\mathrm{e}^{4M_1(t-s)}\mathrm{d}s\right],$$

于是 z^ε 关于 ε 一致有界.同时注意到

$$E[\|x^\varepsilon - \bar{x}\|^2] = E[\|z^\varepsilon\|^2\varepsilon],$$

从而得到

$$\sup_{t\in[0,T]} E[\|x^\varepsilon - \bar{x}\|^2] \leqslant K\varepsilon \to 0(\varepsilon \to 0),$$

即 $x^\varepsilon \to \bar{x}$.同理可证,$z^\varepsilon \to z$.证毕.

同理,可得到如下引理.

引理 4　记 $z_i^\varepsilon = \dfrac{x_i^\varepsilon - \bar{x}_i}{\varepsilon}, i=1,2$,则有 $z^\varepsilon \to z$,其中 (z_1,z_2) 满足如下方程

$$\begin{cases}\mathrm{d}z_1 = r_1z_1 - 2a_{11}\bar{x}_1z_1 - a_{12}\bar{x}_2z_1 - \bar{h}_1z_1 - a_{12}\bar{x}_1z_2 - \bar{x}_1\Delta h_1 + \sigma_1z_1\mathrm{d}B_1 + \displaystyle\int_B c_1(u)z_1\tilde{N}(\mathrm{d}t,\mathrm{d}u),\\ \mathrm{d}z_2 = -a_{21}\bar{x}_2z_1 + r_2z_2 - a_{21}\bar{x}_1z_2 - 2a_{22}\bar{x}_2z_2 - \bar{h}_2z_2 - \bar{x}_2\Delta h_2 + \sigma_2z_2\mathrm{d}B_2 + \displaystyle\int_B c_2(u)z_2\tilde{N}(\mathrm{d}t,\mathrm{d}u),\\ z_1(0) = z_2(0) = 0.\end{cases}$$

下面建立本文的主要结果.

定理 1　设 $\bar{h} = (\bar{h}_1,\bar{h}_2)$ 为最优控制问题(1)-(3)的最优收获策略,$\bar{x} = (\bar{x}_1,\bar{x}_2)$ 为对应的最优状态,则有

$$(\mathrm{e}^{-\lambda t} - p_1(t))\bar{x}_1(t)(h_1 - \bar{h}_1) + (\mathrm{e}^{-\lambda t} - p_2(t))\bar{x}_2(t)(h_2 - \bar{h}_2) \leqslant 0,$$

$t \in [0,T]$, $\forall h = [h_1,h_2] \in \Lambda$.

证明　定义哈密尔顿泛函如下

$$H(t,x,h,p,q,r) = \mathrm{e}^{-\lambda t}h_1x_1 + \mathrm{e}^{-\lambda t}h_2x_2 + x_1(r_1 - h_1 - a_{11}x_1 - a_{12}x_2)p_1 + \sigma_1x_1q_{11} + x_2(r_2 - h_2 - a_{21}x_1 - a_{22}x_2)p_2 + \sigma_2x_2q_{22} + \int_B c_1(u)x_1r_1(t,u)\lambda(\mathrm{d}u) + \int_B c_2(u)x_2r_2(t,u)\lambda(\mathrm{d}u),$$

其共轭方程为

$$\begin{cases}\mathrm{d}p_1 = -\left(\mathrm{e}^{-\lambda t}\bar{h}_1 + (r_1 - \bar{h}_1 - 2a_{11}\bar{x}_1 - a_{12}\bar{x}_2)p_1 - a_{21}\bar{x}_2p_2 + \sigma_1q_{11} + \displaystyle\int_B c_1(u)r_1\lambda(\mathrm{d}u)\right)\mathrm{d}t + q_{11}\mathrm{d}B_1 + q_{12}\mathrm{d}B_2 + \displaystyle\int_B r_1(t^-,u)\tilde{N}(\mathrm{d}t,\mathrm{d}u),\\ \mathrm{d}p_2 = -\left(\mathrm{e}^{-\lambda t}\bar{h}_2 + (r_2 - \bar{h}_2 - a_{21}\bar{x}_1 - 2a_{22}\bar{x}_2)p_2 - a_{12}\bar{x}_1p_1 + \sigma_2q_{22} + \displaystyle\int_B c_2(u)r_2\lambda(\mathrm{d}u)\right)\mathrm{d}t + q_{21}\mathrm{d}B_1 + q_{22}\mathrm{d}B_2 + \displaystyle\int_B r_2(t^-,u)\tilde{N}(\mathrm{d}t,\mathrm{d}u),\\ p_1(T) = p_2(T) = 0,\end{cases}$$

从而有

$$\begin{aligned}\mathrm{d}z_1(t)p_1(t) = {}& p_1\{[(r_1 - \bar{h}_1 - 2a_{11}\bar{x}_1 - a_{12}\bar{x}_2)z_1 - a_{12}\bar{x}_1z_2 - \bar{x}_1\Delta h_1]\mathrm{d}t + \sigma_1z_1\mathrm{d}B_1\} + \\ & z_1\{[-(r_1 - \bar{h}_1 - 2a_{21}\bar{x}_1 - a_{12}\bar{x}_2)p_1 + a_{21}\bar{x}_2p_2 - \mathrm{e}^{-\lambda t}\bar{h}_1 - \sigma_1q_{11} - \\ & \int_B c_1(u)r_1\lambda(\mathrm{d}u)]\mathrm{d}t + q_{11}\mathrm{d}B_1 + q_{12}\mathrm{d}B_2\} + \\ & \sigma_1z_1q_{11}\mathrm{d}t + \int_B c_1(u)r_1z_1\lambda(\mathrm{d}u)\mathrm{d}t + \\ & \int_B(z_1r_1 + c_1z_1p_1 + c_1z_1r_1)\tilde{N}(\mathrm{d}t,\mathrm{d}u) = \\ & (-a_{12}\bar{x}_1p_1z_2 + a_{21}\bar{x}_2p_2z_1 - p_1\bar{x}_1\Delta h_1 - z_1\mathrm{e}^{-\lambda t}\bar{h}_1)\mathrm{d}t + (\sigma_1z_1p_1 + q_{11}z_1)\mathrm{d}B_1 + \\ & q_{12}z_1\mathrm{d}B_2 + \int_B(z_1r_1 + c_1z_1p_1 + c_1z_1r_1)\tilde{N}(\mathrm{d}t,\mathrm{d}u),\end{aligned}$$

$$\begin{aligned}\mathrm{d}z_2(t)p_2(t) = {}& p_2(t)\{[-a_{21}\bar{x}_2z_1 + (r_2 - \bar{h}_2 - a_{21}\bar{x}_1 - 2a_{22}\bar{x}_2)z_2 - \bar{x}_2\Delta h_2]\mathrm{d}t + \sigma_2z_2\mathrm{d}B_2\} + \\ & z_2\{[a_{12}\bar{x}_1p_1 - (r_2 - \bar{h}_2 - a_{21}\bar{x}_1 - 2a_{22}\bar{x}_2)p_2 - \mathrm{e}^{-\lambda t}\bar{h}_2 - \sigma_2q_{22} - \\ & \int_B c_2(u)r_1\lambda(\mathrm{d}u)]\mathrm{d}t + q_{21}\mathrm{d}B_1 + q_{22}\mathrm{d}B_2\} + \\ & \sigma_2z_2q_{22}\mathrm{d}t + \int_B c_2(u)r_2(t^-,u)\lambda(\mathrm{d}u) + \\ & \int_B(z_2r_2 + c_2z_2p_2 + c_2z_2r_2)\tilde{N}(\mathrm{d}t,\mathrm{d}u) = \\ & (-a_{21}\bar{x}_2p_2z_1 + a_{12}\bar{x}_1p_1z_2 - p_2\bar{x}_2\Delta h_2 -\end{aligned}$$

$$z_2 e^{-\lambda t}\bar{h}_2)\mathrm{d}t+(r_2 z_2 p_2+q_{22}z_2)\mathrm{d}B_2+q_{21}\mathrm{d}B_1+\int_B (z_2 r_2+c_2 z_2 p_2+c_2 z_2 r_2)\tilde{N}(\mathrm{d}t,\mathrm{d}u).$$

再注意到 z 和 p 的边界条件,易得

$$0=E[z(T)p(T)-z(0)p(0)]=-E[\int_0^T(\bar{x}_1p_1\Delta h_1+\bar{x}_2p_2\Delta h_2)\mathrm{d}t]-E[\int_0^T e^{-\lambda t}(z_1\bar{h}_1+z_2\bar{h}_2)\mathrm{d}t].$$

由于 $\bar{h}$ 为最优收获,从而有 $J(h^\varepsilon)\leqslant J(\bar{h})$,即

$$\int_0^T e^{-\lambda t}(h_1^\varepsilon(x_1^\varepsilon-\bar{x}_1)+\bar{x}_1(h_1^\varepsilon-\bar{h}_1)+h_2^\varepsilon(x_2^\varepsilon-\bar{x}_2)+\bar{x}_2(h_2^\varepsilon-\bar{h}_2))\mathrm{d}t\leqslant 0.$$

两边同时除以 ε,并取 $\varepsilon\to 0$ 得

$$E[\int_0^T e^{-\lambda t}(z_1\bar{h}_1+z_2\bar{h}_2)\mathrm{d}t]+E[\int_0^T e^{-\lambda t}(\bar{x}_1\Delta\bar{h}_1+\bar{x}_2\Delta\bar{h}_2)\mathrm{d}t]\leqslant 0.$$

将

$$E[\int_0^T e^{-\lambda t}(z_1\bar{h}_1+z_2\bar{h}_2)\mathrm{d}t]=-E[\int_0^T(\bar{x}_1p_1\Delta h_1+\bar{x}_2p_2\Delta h_2)\mathrm{d}t],$$

代入上式并注意到 $\Delta h_1=h_1-\bar{h}_1,\Delta h_2=h_2-\bar{h}_2$, $\forall h=[h_1,h_2]\in\Lambda$,得

$$E[\int_0^T(e^{-\lambda t}-p_1(t))\bar{x}_1(t)(h_1-\bar{h}_1)\mathrm{d}t]+E[\int_0^T(e^{-\lambda t}-p_2(t))\bar{x}_2(t)(h_2-\bar{h}_2)\mathrm{d}t]\leqslant 0,$$

从而有

$$(e^{-\lambda t}-p_1(t))\bar{x}_1(t)(h_1-\bar{h}_1)+(e^{-\lambda t}-p_2(t))\bar{x}_2(t)(h_2-\bar{h}_2)\leqslant 0,$$

$t\in[0,T]$, $\forall h=[h_1,h_2]\in\Lambda$.

4 结语

本文基于变分方法和对偶原理建立了一类随机竞争系统的最优收获策略的必要条件,相应结果的数值实现将进一步研究.在后续的研究工作中,将进一步讨论具有时滞及脉冲效应的种群生态系统的最优收获问题.

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