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Dynamical Analysis for an Impulsive Vaccination Delayed SEIRS Epidemic Model

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Abstract: A delayed SEIRS epidemic model with impulsive vaccination and varying total population size was studied. The results show that there exists an infection-free periodic solution which is globally attractive if $R_1 < 1$, and the disease is permanent if $R_2 > 1$.

Key words: impulsive vaccination; time delay; global attractivity; permanence

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一类脉冲接种时滞SEIRS传染病模型的动力学分析

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摘 要: 研究了一类脉冲接种和总人口变化的时滞SEIRS传染病模型. 结果显示，当 $R_1 < 1$ 时无病周期解是全局吸引的，当 $R_2 > 1$ 时疾病是持续的.

关键词: 脉冲接种；周期解；全局吸引；持续

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0 Introduction

In recent years, epidemic mathematical models have been studied by many scholars. Many infectious diseases in nature incubate inside the hosts for a period of time before the hosts become infectious. Using a compartmental approach, one may assume that a susceptible individual first goes through a latent period after infection and before becoming infectious. The resulting models are of SEIR or SEIRS types, respectively, depending on whether the acquired immunity is permanent or otherwise. Time delay and impulse are introduced into epidemic models, which greatly enrich biologic background, there were some literatures [1-3] about delay epidemic models with impulsive effect.

This paper considers the following model:

$$\begin{cases} \left.\begin{aligned} \dot{S} &= A - \mu S - \beta \frac{I(t)S(t)}{1+\alpha I^2(t)} - (1-p)\mu I(t) + re^{-\mu\omega'}I(t-\omega'), \\ \dot{E} &= \beta \frac{I(t)S(t)}{1+\alpha I^2(t)} - \beta \frac{I(t-\omega)S(t-\omega)}{1+\alpha I^2(t-\omega)} - \mu E + (1-p)\mu I(t), \\ \dot{I} &= \beta \frac{I(t-\omega)S(t-\omega)}{1+\alpha I^2(t-\omega)} - (r+\mu+d)I(t), \\ \dot{R} &= rI(t) - re^{-\mu\tau}I(t-\omega') - \mu R, \end{aligned}\right\} & t \neq n\tau, \\ \left.\begin{aligned} S(n\tau^+) &= (1-\delta)S(n\tau^-), \\ E(n\tau^+) &= E(n\tau^-), \\ I(n\tau^+) &= I(n\tau^-), \\ R(n\tau^+) &= R(n\tau^-) + \delta S(n\tau^-), \end{aligned}\right\} & t = n\tau,\ n \in \mathbf{N}_+, \end{cases} \tag{1}$$

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where all coefficients are positive constants,

$$N(t) = S(t) + E(t) + I(t) + R(t)$$

denotes the total population at time t. $\beta > 0$ is the transmission coefficient. The death rate for disease and physical disease rate are d and μ, respectively. r is the recovery rate of infectious individual. $\delta(0 < \delta < 1)$ is the proportion of those vaccinated successfully at times $t = k\tau, k \in \mathbf{N}_+$. The time delay ω is the latent period of disease. $p\mu I$ $(0 < p < 1)$ is the number of newborns of infectious who transfer to the susceptible class, and $(1-p)\mu I$ is the number of newborns of infectious who are infected. The temporary immunity period of the recovered is ω' and the death of the recovered individuals during the temporary immunity period is the term $re^{-r\omega'}I(t-\omega')$.

The total population size $N(t)$ can be determined by the differential equation

$$\dot{N}(t) = A - \mu N(t) - dI.$$

It follows that

$$\frac{A}{\mu + d} \leqslant \liminf_{t\to\infty} N(t) \leqslant \limsup_{t\to\infty} N(t) \leqslant \frac{A}{\mu}.$$

Note that the variables E and R do not appear in the first and third equations of system (1). This allows us to consider the following subsystem:

$$\begin{cases}\left.\begin{aligned}\dot{S} &= A - \mu S - \beta\frac{I(t)S(t)}{1+\alpha I^2(t)} - \\ &\quad (1-p)\mu I(t) + re^{-\mu\omega'}I(t-\omega'), \\ \dot{I} &= \beta\frac{I(t-\omega)S(t-\omega)}{1+\alpha I^2(t-\omega)} - (r+\mu+d)I(t),\end{aligned}\right\} t \neq n\tau, \\ \left.\begin{aligned}S(n\tau^+) &= (1-\delta)S(n\tau^-), \\ I(n\tau^+) &= I(n\tau^-),\end{aligned}\right\} \begin{aligned} &t = n\tau, \\ &n \in \mathbf{N}_+.\end{aligned}\end{cases} \tag{2}$$

Set $\theta = \max\{\omega, \omega'\}$, the initial condition of system (2) is given as

$$(\varphi_1(\zeta), \varphi_2(\zeta)) \in C_+ = C([-\theta, 0], \mathbf{R}_+^2),$$
$$\varphi_i(0) > 0, i = 1,2. \tag{3}$$

From biological considerations, we discuss system (2) in the closed set

$$\Omega = \{(S,I) \in \mathbf{R}^2 \mid S \geqslant 0, I \geqslant 0, S + I \leqslant \frac{A}{\mu}\}.$$

It is easy to show that Ω is positively invariant with respect to (2).

Lemma 1[4] Considering the following equation:

$$\dot{x}(t) = a_1 x(t-\omega) - a_2 x(t),$$

where $a_1, a_2, \omega > 0$ for $-\omega \leqslant t \leqslant 0$, it follows that

(i) if $a_1 < a_2$, then $\lim_{t\to\infty} x(t) = 0$,

(ii) if $a_1 > a_2$, then $\lim_{t\to\infty} x(t) = \infty$.

1 Global attractivity of infection-free periodic solution

Firstly, we give some basic properties of the following subsystem of system (2):

$$\begin{cases}\dot{S}(t) = A - \mu S(t), t \neq n\tau, \\ S(t^+) = (1-\delta)S(t), t = n\tau, n \in \mathbf{N}_+.\end{cases} \tag{4}$$

We know that the periodic solution of system (4):

$$\tilde{S}_e(t) = \frac{A}{\mu}\left(1 - \frac{\delta}{1-(1-\delta)e^{-\mu\tau}}e^{-\mu(t-k\tau)}\right),$$
$$k\tau < t \leqslant (k+1)\tau, \tag{5}$$

is globally asymptotically stable.

Theorem 1 If $R_1 < 1$, then the infection-free periodic solution $(\tilde{S}_e(t), 0)$ of system (2) is globally attractive, where

$$R_1 = \beta e^{-\mu\omega}\frac{A\mu + Are^{-\mu\omega'}}{\mu^2(d+r+\mu)} \cdot \frac{1-e^{-\mu\tau}}{1-(1-\delta)e^{-\mu\tau}}.$$

Proof Since $R_1 < 1$, then we can choose $\varepsilon_1 > 0$ sufficiently small such that

$$\beta e^{-\mu\omega}\left(\frac{A\mu + Are^{-\mu\omega'}}{\mu^2} \cdot \frac{1-e^{-\mu\tau}}{1-(1-\delta)e^{-\mu\tau}} + \varepsilon_1\right) <$$
$$d + r + \mu, k\tau < t \leqslant (k+1)\tau. \tag{6}$$

From the first equation of system (2), we have

$$\dot{S}(t) < A + \frac{Are^{-\mu\omega'}}{\mu} - \mu S(t).$$

Then we consider the comparison system

$$\begin{cases}\dot{x}(t) = A + \frac{Are^{-\mu\omega'}}{\mu} - \mu x(t), t \neq n\tau, \\ x(t^+) = (1-\delta)x(t), \quad t = n\tau.\end{cases} \tag{7}$$

By computing, we can obtain the unique periodic solution of system (7):

$$\tilde{x}_e(t) = \frac{A\mu + rAe^{-\mu\omega'}}{\mu^2}\left(1 - \frac{\delta}{1-(1-\delta)e^{-\mu\tau}}e^{-\mu(t-k\tau)}\right),$$
$$k\tau < t \leqslant (k+1)\tau,$$

which is globally asymptotically stable.

Let $(S(t), I(t))$ be the solution of system (2) with initial values (3), $x(t)$ be the solution of

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system (4) with initial value $x(0^+) = S_0$. There exists an integer $k_1 > 0$ such that

$$S(t) < \tilde{x}_e(t) + \varepsilon_1, k\tau < t \leqslant (k+1)\tau,$$

that is

$$S(t) < \tilde{S}_e(t) + \varepsilon_1 \leqslant \frac{A\mu + rAe^{-\mu\omega'}}{\mu^2} \cdot \frac{1 - e^{-\mu\tau}}{1 - (1-\theta)e^{-\mu\tau}} + \varepsilon_1 \triangleq \tilde{S}, \quad (8)$$

where $k\tau < t \leqslant (k+1)\tau, k > k_1, \tilde{S}_e(t)$ is defined in (5). From the second equation of system (2), we know that (8) implies that

$$\dot{I}(t) \leqslant \beta e^{-\mu\omega} \tilde{S} I(t-\omega) - (r+\mu+d) I(t), t > k_1\tau + \omega.$$

Consider the following comparison system:

$$\dot{y}(t) = \beta e^{-\mu\omega} \tilde{S} y(t-\omega) - (r+\mu+d) y(t), t > k_1\tau + \omega. \quad (9)$$

From (6), we have $\beta e^{-\mu\omega}\tilde{S} < r + \mu + d$. In view of Lemma 1, we have $\lim_{t\to\infty} y(t) = 0$.

Let $y(t)$ be the solution of system (9) with initial value $y(\zeta) = \phi(\zeta) > 0 (\zeta \in [-\omega, 0])$. We have $\limsup_{t\to\infty} I(t) \leqslant \limsup_{t\to\infty} y(t) = 0$. Incorporating into the positivity of $I(t)$, we know that $\lim_{t\to\infty}(t) = 0$. Therefore, for any $\varepsilon_2 > 0$ (sufficiently small), there exists an integer $k_2 > k_1$ such that $I(t) < \varepsilon_2$ for all $t > k_2\tau$.

For the first equation of system (2), we have

$$\dot{S}(t) \geqslant -\beta\varepsilon_2 S(t) + A - \mu S(t) - (1-p)\mu\varepsilon_2.$$

Considering the comparison impulsive different equation for $t > k_2\tau, k > k_2$,

$$\begin{cases} \dot{z}(t) = A - (1-p)\mu\varepsilon_2 - (\beta\varepsilon_2 + \mu) z(t), & t \neq n\tau, \\ z(t^+) = (1-\delta) z(t), & t = n\tau, \end{cases} \quad (10)$$

we have the unique periodic solution of system (10):

$$\tilde{z}_e(t) = \frac{A - (1-p)\mu\varepsilon_2}{\beta\varepsilon_2 + \mu} + \left(z^* - \frac{A - (1-p)\mu\varepsilon_2}{\beta\varepsilon_2 + \mu}\right) e^{-\mu(t-k\tau)}, \quad k\tau < t \leqslant (k+1)\tau,$$

which is globally asymptotically stable, where

$$z^* = \frac{(A - (1-p)\mu\varepsilon_2)(1-\delta)(1-e^{-\mu\tau})}{(\beta\varepsilon_2 + \mu)(1 - (1-\delta)e^{-\mu\tau})}.$$

Let $z(t)$ be the solution of system (10) with initial values $z(0^+) = S_0$. There exists an integer $k_3 > k_2$ such that

$$S(t) > \tilde{z}_e(t) - \varepsilon_2, k\tau < t \leqslant (k+1)\tau, k > k_3. \quad (11)$$

Because ε_1 and ε_2 are arbitrary small, it follows from (8) and (11) that

$$\tilde{S}_e(t) = \frac{A}{\mu}\left(1 - \frac{\delta}{1-(1-\delta)e^{-\mu\tau}} e^{-\mu(t-k\tau)}\right), \quad k\tau < t \leqslant (k+1)\tau,$$

is globally attractive. Therefore, the infection-free solution $(\tilde{S}(t), 0)$ is globally attractive. This completes the proof.

2 Permanence

Denote

$$R_2 = \frac{\mu A\beta e^{-\mu\omega}}{(r+\mu+d)(\mu^2 + \alpha A^2)} \cdot \frac{(1-\delta)(1-e^{-\mu\tau})}{(1-(1-\delta)e^{-\mu\tau})},$$

$$I^* = \frac{\mu^2 + \alpha A^2}{\beta}(R_2 - 1).$$

Theorem 2 Suppose $R_2 > 1$, then there is a positive constant q such that each positive solution $(S(t), I(t))$ of system (2) satisfies $I(t) \geqslant q$ if t is large enough.

Proof Suppose $(S(t), I(t))$ is any positive solution of system (2) with initial conditions (3). The second equation of system (2) may be rewritten as

$$\dot{I}(t) = \beta e^{-\mu\omega} \frac{S(t) I(t)}{1 + \alpha I^2(t)} - (r+\mu+d) I(t) - \beta e^{-\mu\omega} \frac{d}{dt}\int_{t-\omega}^{t} \frac{S(\theta) I(\theta)}{1 + \alpha I^2(\theta)} d\theta. \quad (12)$$

Define

$$V(t) = I(t) + \beta e^{-\mu\omega} \frac{d}{dt}\int_{t-\omega}^{t} \frac{S(\theta) I(\theta)}{1 + \alpha I^2(\theta)} d\theta.$$

Calculating the derivative of $V(t)$ along the solution of system (2), it follows from (12) that

$$\dot{V}(t) = \beta e^{-\mu\omega} \frac{S(t) I(t)}{1 + \alpha I^2(t)} - (r+\mu+d) I(t). \quad (13)$$

Since $R_2 > 1$, we easily see that $I^* > 0$ and there exists sufficiently small $\zeta > 0$ such that

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$$\frac{\beta\mu^2 e^{-\mu\omega}}{(r+\mu+d)(\mu^2+\alpha A^2)}(h-\zeta) > 1,$$

where

$$h = \frac{(A-(1-p)\mu I^*)(1-\delta)(1-e^{-\mu\tau})}{(\beta I^*+\mu)(1-(1-\delta)e^{-\mu\tau})}. \tag{14}$$

Suppose that there is a $t_0 > 0$ such that $I(t) < I^*$ for all $t > t_0$, it follows from the first equation of (2) that, for $t > t_0$,

$$\dot{S}(t) \geqslant A - \beta I^* S(t) - \mu S(t) - (1-p)\mu I^*.$$

Consider the following comparison impulsive system for $t \geqslant t_0$,

$$\begin{cases}\dot{u}(t) = A - \beta I^* u(t) - \\ \qquad \mu u(t) - (1-p)\mu I^*, & t \neq n\tau, \\ u(t^+) = (1-\delta)u(t), & t = n\tau.\end{cases} \tag{15}$$

We obtain that

$$\tilde{u}_e(t) = \frac{A-(1-p)\mu I^*}{\beta I^*+\mu} + \left(u^* - \frac{A-(1-p)\mu I^*}{\beta I^*+\mu}\right)e^{-\mu(t-k\tau)},$$
$$k\tau < t(k+1) \leqslant \tau,$$

which is globally asymptotically stable, where

$$u^* = \frac{(A-(1-p)\mu I^*)(1-\delta)(1-e^{-\mu\tau})}{(\beta I^*+\mu)(1-(1-\delta)e^{-\mu\tau})}.$$

We know that there exists a $t_1 (t_1 > t_0 + \omega)$ such that the following inequality holds for $t \geqslant t_1$,

$$S(t) \geqslant \tilde{u}_e(t) - \zeta. \tag{16}$$

Thus $S(t) > u^* - \zeta \triangleq \sigma$ for $t > t_1$, from (14) we have

$$\frac{\beta\mu^2 e^{-\mu\omega}\sigma}{(r+\mu+d)(\mu^2+\alpha A^2)} > 1,$$

by (13) and (15), we have

$$\dot{V}(t) \geqslant (r+\mu+d)\left(\frac{\mu^2\beta e^{-\mu\omega}\sigma}{(r+\mu+d)(\mu^2+\alpha A^2)} - 1\right)I(t), t > t_1. \tag{17}$$

Set

$$I_l = \min_{t\in[t_1, t_1+\omega]} I(t),$$

we will show that $I(t) \geqslant I_l$ for all $t > t_1$. Suppose the contrary, there is a $T_0 > 0$ such that $I(t) \geqslant I_l$ for $t_1 \leqslant t \leqslant t_1 + T_0 + \omega$, $I(t_1 + T_0 + \omega) = I_l$ and

$$\dot{I}(t_1 + T_0 + \omega) \leqslant 0.$$

However, the second equation of system (2) and (16) imply that

$$\dot{I}(t_1+T_0+\omega) \leqslant \left(\frac{\beta\mu^2 e^{-\mu\omega}}{\mu^2+\alpha A^2}S(t_1+T_0) - (r+\mu+d)\right)I_l >$$
$$\left(\frac{\beta\mu^2 e^{-\mu\omega}\sigma}{\mu^2+\alpha A^2} - (r+\mu+d)\right)I_l > 0.$$

This is a contradiction, thus $I(t) \geqslant I_l$ for all $t > t_1$. As a consequence, (17) leads to

$$\dot{V} \geqslant \left(\frac{\beta\mu^2 e^{-\mu\omega}\sigma}{\mu^2+\alpha A^2} - (r+\mu+d)\right)I_l$$

for $t > t_1$, which implies that $V(t) \to \infty$ as $t \to \infty$. This contradicts

$$V(t) \leqslant \frac{A}{\mu} + \omega\beta\frac{A^2}{\mu^2+\alpha A^2}e^{-\mu\omega}.$$

Hence, for any $t_0 > 0$, it is impossible that $I(t) < I^*$ for all $t > t_0$. Following, we are left to consider two cases:

(i) $I(t) \geqslant I^*$ for all large t.

(ii) $I(t)$ oscillates about I^* for all t large enough.

Finally, we will show that

$$I(t) \geqslant I^* e^{-(r+\mu+d)(\tau+T^*)} \triangleq q$$

as t is large sufficiently. Evidently, we only need consider the case (ii).

Let t_1 and t_2 be large sufficiently and satisfy $I(t_1) = I(t_2) = I^*$, $I(t) < I^*$ as $t \in (t_1, t_2)$. If $t_2 - t_1 \leqslant T^* + \omega$, then $\dot{I}(t) \geqslant -(r+\mu+d)I(t)$ and $I(t) = I^*$ imply $I(t) \geqslant q$ for $t \in [t_1, t_2]$. If $t_2 - t_1 > T^* + \omega$, then it is clear that $I(t) \geqslant q$ for all $t \in [t_1, t_1 + T^* + \omega]$. Thus proceeding exactly as the proof for claim, we see that $S(t) > \sigma$ for all $t \in [t_1 + T^*, t_2]$. Next, we will prove that $I(t) \geqslant q$ for all $t \in [t_1, t_1 + T^* + \omega + T_1]$, $I(t_1 + T^* + \omega + T_1) = q$ and $\dot{I}(t_1 + T^* + T_1 + \omega) \leqslant 0$. Using the second equation of system (2) as $t = t_1 + T^* + T_1 + \omega$, we further obtain that

$$\dot{I}(t) \geqslant \frac{\beta e^{-\mu\omega}\mu^2}{\mu^2+\alpha A^2}S(t-\tau)I(t-\tau) - (r+\mu+d)I(t) \geqslant$$
$$\left(\frac{\beta e^{-\mu\omega}\mu^2\sigma}{\mu^2+\alpha A^2} - (r+\mu+d)\right)q > 0,$$

which is a contradiction. So, $I(t) \geqslant q$ is valid for all $t \in [t_1, t_2]$. This completes the proof.

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Theorem 3 System (2) is permanent provided $R_2 > 1$.

Proof Denote $(S(t), I(t))$ be the solution of system (2). From the first equation of system (2), we have

$$\dot{S}(t) \geqslant pA - \frac{\mu^2 + \beta A}{\mu} S(t).$$

By the similar argument as those in the proof of Theorem 1, we have $\lim_{t\to\infty} S(t) \geqslant q_*$, where

$$q_* = \frac{pA\mu(1-\delta)(1-e^{-\mu\tau})}{(\mu^2+\beta A)(1-(1-\delta)e^{-\mu\tau})} - \varepsilon_0,$$

ε_0 is sufficiently small.

We let $\Omega_0 = \{(S, I): q_* \leqslant S, q \leqslant I, S + I \leqslant \frac{A}{\mu}\}$. We know that Ω_0 is a global attractor in Ω, every solution of system (2) with initial conditions (3) will eventually enter and remain in region Ω_0. Therefore, system (2) is permanent. This completes the proof.

3 Discussions

We have analyzed the latent period and impulsive vaccination bring effects on system (2). Theorem 1 and Theorem 2 imply that the disease dynamics of (2) is completely determined by R_1 and R_2. Hence, the vaccination effects depend on whether R_1 can be reduced to be below unity or not. We can obtain that $\frac{\partial R_1}{\partial \theta} < 0$ and $\frac{\partial R_1}{\partial \tau} > 0$, which show pulse vaccination strategy always has a good effect for disease control by decreasing R_1.

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